A Local Trace Formula and the Multiplicity One Theorem for the Ginzburg-Rallis Model

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I would like to dedicate this paper to my parents.

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Abstract

Following the method developed by Waldspurger and Beuzart-Plessis in their proof of the local Gan-Gross-Prasad conjecture, we are able to prove a local trace formula for the Ginzburg-Rallis model. Then by applying that trace formula, we prove a multiplicity formula for the Ginzburg-Rallis model for tempered representations. Using that multiplicity formula, we prove the multiplicity one theorem for all tempered L-packets. In some cases, we also proved the epsilon dichotomy conjecture which gives a relation between the multiplicity and the exterior cube epsilon factor. Finally, in the archimedean case, we proved some partial results for the general generic representations by applying the open orbit method.

Contents

Chapter 1

Introduction and the Main Results

1.1 The Ginzburg-Rallis Models

D. Ginzburg and S. Rallis found in their paper ([GR00]) a global integral representation for the partial exterior cube L-function $L^S(s, \pi, \wedge^3)$ attached to any irreducible cuspidal automorphic representation π of $\mathrm{GL}_6(\mathbb{A})$. By using the regularized Siegel-Weil formula of Kudla and Rallis([KR94]), they discovered that the nonvanishing of the central value of the partial exterior cube L-function $L^S(\frac{1}{2}, \pi, \wedge^3)$ is closely related to the Ginzburg-Rallis period, which will be defined as follows. The relation they discovered is similar to the global Gan-Gross-Prasad conjecture ([GP92], [GP94], [GGP12]), but for a different setting.

Let k be a number field, $\mathbb{A}$ be the ring of adeles of k. Take $P = P_{2,2,2} = MU$ be the standard parabolic subgroup of $G = \mathrm{GL}_6$ whose Levi part M is isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, and whose unipotent radical U consists of elements of the form

$$u = u(X, Y, Z) := \begin{pmatrix} I_2 & X & Z \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix}. \tag{1.1}$$

We define a character ξ on U by

$$\xi(u(X, Y, Z)) := \psi(a\mathrm{tr}(X) + b\mathrm{tr}(Y)) \tag{1.2}$$

where ψ is a non-trivial additive character on $k\backslash\mathbb{A}$, and $a, b \in \mathbb{A}^\times$.

It's clear that the stabilizer of ξ is the diagonal embedding of GL_2 into M, which is denoted by H. For a given idele character χ of $\mathbb{A}^\times/k^\times$, one induces a one dimensional representation ω of $H(\mathbb{A})$ given by $\omega(h) := \chi(\det(h))$, which is clearly trivial when restricted to $H(k)$. Now the character ξ can be extended to the semi-direct product

$$R := H \ltimes U \tag{1.3}$$

by making it trivial on H. Similarly we can extend the character ω to R. It follows that the one dimensional representation $\omega \otimes \xi$ of $R(\mathbb{A})$ is well defined and it is trivial when restricted to the k-rational points $R(k)$. Then the Ginzburg-Rallis period for any cuspidal automorphic form ϕ on $\mathrm{GL}_6(\mathbb{A})$ with central character χ^2 is defined to be

$$\mathcal{P}_{R,\omega\otimes\xi}(\phi) = \int_{H(k)Z_G(\mathbb{A})\backslash H(\mathbb{A})} \int_{U(k)\backslash U(\mathbb{A})} \phi(hu)\xi^{-1}(u)\omega^{-1}(h)dudh. \tag{1.4}$$

As in the Jacquet conjecture for the trilinear period of GL_2 ([HK04]) and in the global Gan-Gross-Prasad conjecture ([GGP12]) more generally, Ginzburg and Rallis find that the central value of the partial exterior cube L-function, $L^S(\frac{1}{2}, \pi, \wedge^3)$ may also be related to the quaternion algebra version of the Ginzburg-Rallis period $\mathcal{P}_{R,\sigma\otimes\xi}$. More precisely, let D be a quaternion algebra over k, and consider $G_D := \mathrm{GL}_3(D)$, a k-inner form of GL_6. In the group G_D, they define

$$H_D = \{h_D = \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix} \mid g \in D^\times\} \tag{1.5}$$

and

$$U_D = \{u_D(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in D\}. \tag{1.6}$$

In this case, the corresponding character ξ_D of U_D is defined in same way except that the trace in the definition of ξ is replaced by the reduced trace of the quaternion algebra D. Similarly, the character ω_D on H_D is defined by using the reduced norm of the quaternion algebra D. Now the subgroup R_D is defined to be the semi-direct product $H_D \ltimes U_D$ and the corresponding one dimensional representation $\omega_D \otimes \xi_D$ of

$R_D(\mathbb{A})$ is well defined. The D-version of the Ginzburg-Rallis period for any cuspidal automorphic form $\phi_D D$ on $\mathrm{GL}_3(D)(\mathbb{A})$ with central character χ^2 is defined to be

$$\mathcal{P}_{R_D,\omega_D\otimes\xi_D}(\phi_D) := \int_{H_D(k)Z_{G_D}(\mathbb{A})\backslash H_D(\mathbb{A})} \int_{U_D(k)\backslash U_D(\mathbb{A})} \phi_D(hu)\xi_D^{-1}(u)\sigma_D^{-1}(h)dudh. \quad (1.7)$$

In [GR00], they form a conjecture on the relation between the periods above and the central value $L^S(\frac{1}{2},\pi,\wedge^3)$.

Conjecture 1.1.1 (Ginzburg-Rallis, [GR00])**.** *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_6(\mathbb{A})$ with central character ω_π. Assume that there exists an idele character χ of $\mathbb{A}^\times/k^\times$ such that $\omega_\pi=\chi^2$. Then the central value $L^S(\frac{1}{2},\pi,\Lambda^3)$ does not vanish if and only if there exists a unique quaternion algebra D over k and there exists the Jacquet-Langlands correspondence π_D of π from $\mathrm{GL}_6(\mathbb{A})$ to $\mathrm{GL}_3(D)(\mathbb{A})$, such that the period $\mathcal{P}_{R_D,\sigma_D\otimes\xi_D}(\phi_D)$ does not vanish for some $\phi_D\in\pi_D$, and the period $\mathcal{P}_{R_{D'},\sigma_{D'}\otimes\xi_{D'}}(\phi_{D'})$ vanishes identically for all quaternion algebra D' which is not isomorphic to D over k, and for all $\phi_{D'}\in\pi_{D'}$.*

It is clear that this conjecture is an analogy of the global Gan-Gross-Prasad conjecture for classical groups ([GGP12]) and the Jacquet conjecture for the triple product L-functions for GL_2, which is proved by M. Harris and S. Kudla in [HK04]. It is also clear that Conjecture 1.1.1 is now a special case of the general global conjecture of Y. Sakellaridis and A. Venkatesh for periods associated to general spherical varieties ([SV]).

Similarly to the Gan-Gross-Prasad model, there is also a local conjecture for the Ginzburg-Rallis model, which is the main result of this paper. The conjecture at local places has been expected since the work of [GR00], and was first discussed in details by Dihua Jiang in his paper [J08]. Now let F be a local field of characteristic zero, D be the unique quaternion algebra over F if $F\neq\mathbb{C}$. Then we may also define the groups H,U,R,H_D,U_D, and R_D as above. The local conjecture can be stated as follows, using the local Jacquet-Langlands correspondence established in [DKV84].

Conjecture 1.1.2 (Jiang, [J08])**.** *For any irreducible admissible representation π of $\mathrm{GL}_6(F)$, let π_D be the local Jacquet-Langlands correspondence of π to $\mathrm{GL}_3(D)$ if it exists, and zero otherwise. In particular, π_D is always zero if $F=\mathbb{C}$. Assume that there exists a character χ of $F^\times$ such that $\omega_\pi=\chi^2$. For a given non-trivial additive character ψ of*

F, similar to the global case, we can define the one dimensional representation $\omega \otimes \xi$ of $R(F)$ and $\omega_D \otimes \xi_D$ of $R_D(F)$, respectively. Then the following identity

$$\dim(\mathrm{Hom}_{R(F)}(\pi, \omega \otimes \xi)) + \dim(\mathrm{Hom}_{R_D(F)}(\pi_D, \omega_D \otimes \xi_D)) = 1 \tag{1.8}$$

holds for all irreducible generic representation π of $\mathrm{GL}_6(F)$.

As in the local Gan-Gross-Prasad conjecture ([GGP12]), Conjecture 1.1.2 can be reformulated in terms of local Vogan packets and the assertion in the conjecture is expressed as the local multiplicity one over local Vogan packets. Here although $\mathrm{GL}_6(F)$ does not have non-trivial pure inner form, as we already make the central character assumption, we are actually working with the pair $(\mathrm{PGL}_6, \mathrm{PGL}_2 \ltimes U)$ which have non-trivial pure inner form. For any quaternion algebra D over F which may be F-split, define

$$m(\pi_D) := m(\pi_D, \sigma_D \otimes \xi_D) := \dim(\mathrm{Hom}_{R_D(F)}(\pi_D, \sigma_D \otimes \xi_D)). \tag{1.9}$$

The local multiplicity one theorem for each individual irreducible admissible representation π_D of $\mathrm{GL}_3(D)$ asserts that

$$m(\pi_D) = m(\pi_D, \sigma_D \otimes \xi_D) \leq 1 \tag{1.10}$$

for any given $\sigma_D \otimes \xi_D$. This local multiplicity one theorem was proved in [N06] over a p-adic local field and in [JSZ11] over an archimedean local field. Then (1.8) becomes

$$m(\pi) + m(\pi_D) = 1.$$

Another aspect of the local conjecture is the so-called *ϵ-dichotomy conjecture*, which relates the multiplicity with the value of the exterior cube epsilon factor. Assume that π is generic and the central character of π is trivial. Then the conjecture can be stated as follows.

Conjecture 1.1.3. *With the assumptions above, the followings hold.*

$$m(\pi) = 1 \iff \epsilon(1/2, \pi, \wedge^3) = 1,$$
$$m(\pi) = 0 \iff \epsilon(1/2, \pi, \wedge^3) = -1.$$

In this paper, we always fix a Haar measure dx on F and an additive character ψ such that the Haar measure is selfdual for Fourier transform with respect to ψ. We use such dx and ψ in the definition of the ϵ factor. For simplicity, we will write the epsilon factor as $\epsilon(s,\pi,\rho)$ instead of $\epsilon(s,\pi,\rho,dx,\psi)$.

Remark 1.1.4. *In Conjecture 1.1.3, we do need the assumption that the central character of π is trivial. Otherwise, the exterior cube of the Langlands parameter of π will no longer be selfdual, and hence the value of the epsilon factor at $1/2$ may not be ± 1.*

Remark 1.1.5. *In the definition of the character ξ, we introduce two coefficients $a,b \in F^{\times}$. It is easy to see that the multiplicity is actually independent of the choice of a and b. The reason we introduce these two coefficients is for the proof of the geometric side of the trace formula from Chapter 9 to Chapter 12. For all the rest chapters, we will just take $a=b=1$.*

1.2 Main Results

The main goal of this paper to prove the local conjectures stated in the previous section for tempered representations. We first talk about our results for Conjecture 1.1.2.

Theorem 1.2.1. *For every tempered representation π of $\mathrm{GL}_6(F)$ with central character χ^2, Conjecture 1.1.2 holds. In particular, we have*

$$m(\pi)+m(\pi_D)=1.$$

Our proof of Theorem 1.2.1 uses Waldspurger's method in his proof of the local Gan-Gross-Prosad conjecture (orthogonal case) in [W10] and [W12]; and also some techniques introduced by Beuzart-Plessis in his proof of the local Gan-Gross-Prosad conjecture (unitary case) in [B12] and [B15]. In the p-adic case, the key ingredient of the proof is a local relative trace formula for the Ginzburg-Rallis model, which will be called the trace formula in this paper for simplicity, unless otherwise specified.

To be specific, let $f \in C_c^{\infty}(Z_G(F)\backslash G(F), \chi^{-2})$ be a strongly cuspidal function (see Section 3.4 for the definition of strongly cuspidal functions), and define the function $I(f,\cdot)$ on $R(F)\backslash G(F)$ to be

$$I(f,x)=\int_{R(F)/Z_G(F)} f(x^{-1}hx)\xi(h)\omega(h)dh.$$

Then define

$$I(f)=\int_{R(F)\backslash G(F)} I(f,g)dg. \tag{1.11}$$

We will prove in Section 8.1 that the integral defining $I(f)$ is absolutely convergent. The distribution in the trace formula is just $I(f)$.

Now we define the spectral and geometric sides of the trace formula. To each strongly cuspidal function $f\in C_c^{\infty}(Z_G(F)\backslash G(F),\chi^{-2})$, one can associate a distribution θ_f on $G(F)$ via the weighted orbital integral (see Section 3.4). It was proved in [W10] that the distribution θ_f is a quasi-character in the sense that for every semisimple element $x\in G_{ss}(F)$, θ_f is a linear combination of the Fourier transform of the nilpotent orbital integrals of $\mathfrak{g}_x$ near x. For each nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}_x$, let $c_{\theta_f,\mathcal{O}}(x)$ be the coefficient, it is called the germ of the distribution θ_f. Let $\mathcal{T}$ be a subset of subtorus of H as defined in Section 5.1. For any $t\in T_{reg}(F)$ and $T\in\mathcal{T}$, define $c_f(t)$ to be $c_{\theta_f,\mathcal{O}_t}(t)$ where $\mathcal{O}_t$ is the unique regular nilpotent orbit in $\mathfrak{g}_t$. For detailed description of $\mathcal{O}_t$, see Section 5.1. Then we define the geometric side of our trace formula to be

$$I_{geom}(f)=\sum_{T\in\mathcal{T}}|W(H,T)|^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)} c_f(t)D^H(t)\Delta(t)\chi(\det(t))dt$$

where $D^H(t)$ is the Weyl determinant and $\Delta(t)$ is some normalized function as defined in Section 5.1. For the spectral side, define

$$I_{spec}(f)=\int_{\Pi_{temp}(G,\chi^2)}\theta_f(\pi)m(\bar{\pi})d\pi$$

where $\Pi_{temp}(G,\chi^2)$ is the set of irreducible tempered representations of $G(F)=\mathrm{GL}_6(F)$ with central character χ^2, $d\pi$ is the measure on $\Pi_{temp}(G,\chi^2)$ as defined in Section 2.9, and $\theta_f(\pi)$ is the weighted character as defined in Section 3.5. Then the trace formula we proved in this paper is just

$$I_{spec}(f)=I(f)=I_{geom}(f). \tag{1.12}$$

The proof of the spectral side of the trace formula will be given in Chapter 8, while the geometric side will be proved in Chapter 12. Similarly, we can also have the quaternion version of the trace formula.

After proving the trace formula, we are going to prove a multiplicity formula for the Ginzburg-Rallis model:

$$m(\pi)=m_{geom}(\pi),\ m(\pi_D)=m_{geom}(\pi_D). \tag{1.13}$$

Here $m_{geom}(\pi)$ (resp. $m_{geom}(\pi_D)$) is defined in the same way as $I_{geom}(f)$ except replacing the distribution θ_f by the distribution character θ_π (resp. θ_{π_D}) associated to the representation π (resp. π_D). For the complete definition of the multiplicity formula, see Section 13.1. Once this formula is proved, we can use the relation between the distribution characters θ_π and θ_{π_D} under the local Jacquet-Langlands correspondence to cancel out all terms in the expression of $m_{geom}(\pi) + m_{geom}(\pi_D)$ except the term $c_{\theta_\pi,\mathcal{O}_{reg}}$, which is the germ at the identity element. Then the work of Rodier ([Rod81]) shows that $c_{\theta_\pi,\mathcal{O}_{reg}} = 0$ if π is non-generic, and $c_{\theta_\pi,\mathcal{O}_{reg}} = 1$ if π is generic. Because all tempered representations of $GL_n(F)$ are generic, we get the following identity

$$m_{geom}(\pi) + m_{geom}(\pi_D) = 1. \tag{1.14}$$

And this proves Theorem 1.2.1. The proof of the multiplicity formula uses the trace formula we mentioned above, together with the Plancherel formula and Arthur's local trace formula. For details, see Chapter 13.

In the archimedean case, although we can use the same method as in the p-adic case (like Beuzart-Plessis did in [B15] for the GGP case), it is actually much easier. All we need to do is to show that the multiplicity is invariant under the parabolic induction, and this will be done in Chapter 6 for both p-adic and archimedean case. Then if $F = \mathbb{R}$, since only $\mathrm{GL}_1(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$ have discrete series, we can reduce the problem to the trilinear GL_2 model which has been considered by D. Prasad in his thesis [P90], and also by Loke in [L01]. If $F = \mathbb{C}$, every generic representation is a principal series, this reduces the problem to the reduced model associated to the torus whose multiplicity is always 1. For details, see Chapter 7.

Now for the epsilon dichotomy conjecture, our results can be stated as follows.

Theorem 1.2.2. *Let π be an irreducible tempered representation of $\mathrm{GL}_6(F)$ with trivial central character.*

1. *If F is archimedean, Conjecture 1.1.3 holds.*
2. *If F is p-adic, and if π is not discrete series or the parabolic induction of some discrete series of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, Conjecture 1.1.3 holds.*

The proof of the archimedean case will be given in Chapter 7, and the p-adic case will be proved in Chapter 13. Our methods is to show that both the multiplicities and the epsilon factor are invariant under the parabolic induction. Then if $F = \mathbb{R}$, we can reduce to the trilinear GL_2 model case, which has already been proved by Prasad and Loke. If $F = \mathbb{C}$, we can show that the multiplicity is always 1 and the epsilon factor is also always equal to 1. This proves the theorem. If F is p-adic, under our assumptions, there are only two possibilities. One is that the representation is induced from P, then we can still reduce to the trilinear GL_2 model case. The other possibility is that the representation is induced from some type II parabolic subgroup (see Section 4.5 for the definition of type II models). In this case, one can show that the multiplicity and the epsilon factor are both equal to 1. This proves the theorem.

Moreover, our methods can also be applied to all reduced models of the Ginzburg-Rallis model coming from the parabolic induction. For some models such results are well known (like the trilinear GL_2 model); but for many other models, as far as we know, such results never appear in literature. The reduced models will be discussed in Section 4.5. The trace formulas and the multiplicity formulas for those models will be discussed in Section 5.4.

After we proved the tempered case, it is naturally to ask how about the general generic representations. In this case, we only have partial result for the archimedean case. Before we state it, we need some preparation.

If $F = \mathbb{C}$, by the Langlands classification, any generic representation π is a principal series. In other word, let $B = M_0 U_0$ be the Borel subgroup consisting of all the lower triangular matrix, here $M_0 = (\mathrm{GL}_1)^6$ is just the group of diagonal matrix. Then π is of the form $I_B^G(\chi)$ where $\chi = \otimes_{i=1}^6 \chi_i$ is a character on M_0 and I_B^G is the normalized parabolic induction. For $1 \leq i \leq 6$, we can find an unitary character σ_i and some real number $s_i \in \mathbb{R}$ such that $\chi_i = \sigma_i |\ |^{s_i}$. Without loss of generality, we assume that $s_i \leq s_j$ for any $i \geq j$. Then if we combine those representations with the same exponents s_i, we can find a parabolic subgroup $Q = LU_Q$ containing B with $L = \times_{i=1}^k \mathrm{GL}_{n_i}$, a representation $\tau = \otimes_{i=1}^k \tau_i |\ |^{t_i}$ of $L(F)$ where τ_i are all tempered and the exponents t_i are strictly increasing (i.e. $t_1 < t_2 < \cdots < t_k$) such that $\pi = I_Q^G(\tau)$. On the other hand, we can also write π as $I_P^G(\pi_0)$ with $\pi_0 = \pi_1 \otimes \pi_2 \otimes \pi_3$ and π_i be the parabolic induction of $\chi_{2i-1} \otimes \chi_{2i}$.

Theorem 1.2.3. *Assume that $F = \mathbb{C}$, with the same assumptions as in Conjecture 1.1.2 and with the notation above, the followings hold.*

1. *If $\bar{P} \subset Q$, Conjecture 1.1.2 and Conjecture 1.1.3 hold. In particular, both conjectures hold for the tempered representations.*

2. *If $Q \subsetneq \bar{P}$ and if π_0 satisfies the condition (40) in [L01], Conjecture 1.1.2 and Conjecture 1.1.3 hold.*

The main ingredient of our methods for Theorem 1.2.3 is the open orbit method, which allows us to reduce our problems to the tempered case or the trilinear GL_2 model case. To be specific, if $\bar{P} \subset Q$, by applying the open orbit method, we can reduce to the model related to the Levi subgroup L. Then after twisting τ by some characters, we only need to deal with the tempered case which has already been proved in the first place. If $Q \subset \bar{P}$, by applying the open orbit method, we reduced ourselves to the trilinear GL_2 model case. Then by applying the work of Loke in [L01], we can prove our result. The extra condition in part (2) of Theorem 1.2.1 also comes from [L01].

It is worth to mention that in Theorem 1.2.3(2), the requirement we made for the parabolic subgroup Q force some types of generalized Jacquet integrals to be absolutely convergent, this allows us to apply the open orbit method. If one can prove such integrals have holomorphic continuation, we can actually remove this requirement. This will be discussed in Chapter 14.

If $F = \mathbb{R}$, again by applying the open orbit method, we will have some partial results about Conjecture 1.1.2 and Conjecture 1.1.3 for general generic representations. To be specific, let π be a irreducible generic representation of $G(F)$ with central character χ^2. By the Langlands classification, there is a parabolic subgroup $Q = LU_Q$ containing the lower Borel subgroup and an essential tempered representation $\tau = \otimes_{i=1}^{k} \tau_i| \ |^{s_i}$ of $L(F)$ with τ_i tempered, $s_i \in \mathbb{R}$ and $s_1 < s_2 < \cdots < s_k$ such that $\pi = I_Q^G(\tau)$. We say Q is nice if $Q \subset \bar{P}$ or $\bar{P} \subset Q$.

Theorem 1.2.4. *With the notations above, the following hold.*

1. *If $\pi_D = 0$, assume that Q is nice, then Conjecture 1.1.2 and Conjecture 1.1.3 hold.*

2. *If $\pi_D \neq 0$, we have*
$$m(\pi) + m(\pi_D) \geq 1.$$

Moreover if the central character of π is trivial (as in Conjecture 1.1.3), we have

$$\epsilon(1/2,\pi,\wedge^3)=1\Rightarrow m(\pi)=1;\ m(\pi)=0\Rightarrow \epsilon(1/2,\pi,\wedge^3)=-1.$$

As in the complex case, the assumption on Q can be removed if we can prove the holomorphic continuation of certain generalized Jacquet integrals. This will also be discussed in Chapter 14.

1.3 Organization of the Paper and Remarks on the Proof

In Chapter 2, we introduce basic notation and convention of this paper. We will also talk about the definitions and some basic facts on weighted orbital integrals, weighted character, intertwining operator and the Harish-Chandra-Plancherel formula. In Chapter 3, we will study quasi-characters and strongly cuspidal functions. For Chapter 2 and 3, we follow [W10] and [B15] closely. We will only include the proof if necessary.

In Chapter 4, we study the analytic and geometric properties of the Ginzburg-Rallis model. In particular, we show that it is a wavefront spherical variety and has polynomial growth as a homogeneous space. This gives us the weak Cartan decomposition for the archimedean case. The p-adic case will be proved in Appendix A by the explicit construction. Then by applying those results, we prove some estimations for various integrals which will be used in later chapters. The proof of some estimations are similar to the GGP case in [B15], we only include the proof here for completion. At the end of Chapter 4, we will also talk about the reduce models of the Ginzburg-Rallis model coming from the parabolic induction.

In Chapter 5, we will state our trace formula. For the geometric side, we will also consider the Lie algebra version of the trace formula, which will be used in the proof. We will also show that in order to prove the geometric side, it is enough to consider the functions with trivial central character. Finally, we will also introduce the trace formulas for the reduced models. By induction, we will assume that the trace formulas for those reduced models hold.

In Chapter 6, we study an explicit element in the Hom space coming from the (normalized) integration of the matrix coefficient. The goal is to prove that the Hom space is nonzero if and only if the explicit operator is nonzero. It is standard to prove such a statement by using the Plancherel formula together with the fact that the nonvanishing property of the explicit operator is invariant under parabolic induction and unramified twist. However, there are two main difficulties in the proof of such a result for the Ginzburg-Rallis models. First, unlike the Gan-Gross-Prasad case, we do have nontrivial center for the Ginzburg-Rallis model. As a result, for many parabolic subgroups of $\mathrm{GL}_6(F)$ (the one which don't have an analogy in the quaternion case, i.e. the one not of type (6), $(4,2)$ or $(2,2,2)$, we will call theses models "type II models"), it is not clear why the nonvanishing property of the explicit operator is invariant under the unramified twist. Instead, we show that for such parabolic subgroups, the explicit operator will always be nonzero.

Another difficulty is that unlike the Gan-Gross-Prasad case, when we do parabolic induction, we don't always have the strongly tempered model (in the GGP case, one can always go up to the codimension one case which is strongly tempered, then run the parabolic induction process). As a result, in order to prove the nonvanishing property of the explicit operator is invariant under parabolic induction, it is not enough to just change the order of the integral. This is because if the model is not strongly tempered, the explicit operator is defined via the normalized integral, not the original integral. We will find a way to deal with this issue in Chapter 6, but we have to treat the p-adic case and the archimedean case separately. For details, see Section 6.3 and 6.4.

In Chapter 7, we prove our main Theorems for the archimedean case by reducing it to the reduced models. Then we need to apply the results of the trilinear GL_2 model due to Prasad and Loke in [P90], .

In Chapter 8, we will prove the spectral side of the trace formula. In the trace formula, we do have a truncated function which is for the proof of the geometric side. In Section 8.1, we first show that the integral defining our distribution $I(f)$ is actually absolutely convergent. This allows us to get rid of the truncated function. We will postpone the proof of a technical proposition (i.e. Proposition 8.1.1) to Appendix B. Then in Section 8.2, we prove the spectral side by applying the results in the previous chapters.

Start from Chapter 9, we are going to prove the geometric side of the trace formula. In Chapter 9, we deal with the localization of the trace formula. The goal of this section is to reduce our problem to the Lie algebra level. In Chapter 10, we study the slice representation of the normal space. As a result, we transfer our integral to the form $\int_{A_T(F)\backslash G(F)}$ where T is some maximal torus of G. The reason we do this is that we want to apply the local trace formula developed by Arthur in [Ar91] as Waldspurger did in [W10]. In Chapter 11, we prove that we are actually able to change our truncation function to the one given by Arthur in his local formula. After this is done, we can apply Arthur's local trace formula to calculate the distribution in our trace formula. More precisely, at beginning, the distribution is a limit of the truncated integral. After applying Arthur's local trace formula, we can calculate that limit explicitly. Finally in Chapter 12, we finish the proof of the trace formula.

It is worth to mention that the proof of the geometric expansion is quiet different from the case of the local Gan-Gross-Prasad conjecture in [W10]. Namely, in their case, the additive character is essentially attached to the simple roots, which is not the case in our situation. This difference leads to the technical complication on the proof of some unipotent invariance. As a result, we have to carefully define our truncated function. This will be discussed in detail in Chapter 5 and 11. Another difference is that in this case we do need to worried about the center of the group, this will be discussed in Chapter 5.

In Chapter 13, by applying the trace formula we proved in previous chapters, we are able to prove a multiplicity formula for tempered representations. By applying that multiplicity formula, we can prove our main Theorem 1.2.1. After it, we will also prove the epsilon dichotomy conjecture for some representations, i.e. Theorem 1.2.2.

In Chapter 14, by applying the open orbit method, together with our results for tempered representations, we can prove some partial results for the generic representations over archimedean field, i.e. Theorem 1.2.3 and Theorem 1.2.4.

There are three appendices of this paper. In Appendix A, we prove the weak Cartan decomposition for the p-adic case by the explicit construction. In Appendix B, we prove Proposition 8.1.1. The proof will be the same as the Gan-Gross-Prasad model case in [B15], we only include the proof here for completion. In Appendix C, we will give a summary about the results for the reduced models. The proof of these results is the same as the Ginzburg-Rallis model case we consider in this paper, so we will skip the details.

Chapter 2

Priliminarites

2.1 Notations and Conventions

Let F be a local field of characteristic zero. If F is a p-adic filed, we fix the algebraic closure $\overline{F}$. Let val_F and $|\cdot|_F$ be the valuation and absolute value on F, $\mathfrak{o}_F$ be the ring of integers of F, and $\mathbb{F}_q$ be the residue field. We fix an uniformizer ϖ_F.

For every connected reductive algebraic group G defined over F, let A_G be the maximal split central torus of G and Z_G be the center of G. We denote by $X(G)$ the group of F-rational characters of G. Define $\mathfrak{a}_G =\mathrm{Hom}(X(G),\mathbb{R})$, and let $\mathfrak{a}_G^\ast = X(G)\otimes_{\mathbb{Z}}\mathbb{R}$ be the dual of $\mathfrak{a}_G$. We define a homomorphism $H_G : G(F)\rightarrow \mathfrak{a}_G$ by $H_G(g)(\chi)=\log(|\chi(g)|_F)$ for every $g\in G(F)$ and $\chi\in X(G)$. Let $\mathfrak{a}_{G,F}$ (resp. $\tilde{\mathfrak{a}}_{G,F}$) be the image of $G(F)$ (resp. $A_G(F)$) under H_G. In the archimedean case, $\mathfrak{a}_G=\mathfrak{a}_{G,F}=\tilde{\mathfrak{a}}_{G,F}$; in the p-adic case, $\mathfrak{a}_{G,F}$ and $\tilde{\mathfrak{a}}_{G,F}$ are lattices in $\mathfrak{a}_G$. Let $\mathfrak{a}_{G,F}^\vee=\mathrm{Hom}(\mathfrak{a}_{G,F},2\pi\mathbb{Z})$ and $\tilde{\mathfrak{a}}_{G,F}^\vee=\mathrm{Hom}(\tilde{\mathfrak{a}}_{G,F},2\pi\mathbb{Z})$. Note that both $\mathfrak{a}_{G,F}^\vee$ and $\tilde{\mathfrak{a}}_{G,F}^\vee$ are zero in the archimedean case; and they are lattices in $\mathfrak{a}_G^\ast$ in the p-adic case. Set $\mathfrak{a}_{G,F}^\ast=\mathfrak{a}_G^\ast/\mathfrak{a}_{G,F}^\vee$, and we can identify $i\mathfrak{a}_{G,F}^\ast$ with the group of unitary unramified characters of $G(F)$ by letting $\lambda(g)=e^{<\lambda,H_G(g)>}$, $\lambda\in i\mathfrak{a}_{G,F}^\ast$, $g\in G(F)$. For a Levi subgroup M of G, let $\mathfrak{a}_{M,0}^\ast$ be the subset of elements in $\mathfrak{a}_{M,F}^\ast$ whose restriction to $\tilde{\mathfrak{a}}_{G,F}$ is zero. Then we can identify $i\mathfrak{a}_{M,0}^\ast$ with the group of unitary unramified characters of $M(F)$ which is trivial on $Z_G(F)$.

Denote by $\mathfrak{g}$ the Lie algebra of G. It is clear that G acts on $\mathfrak{g}$ by the adjoint action. Since the Ginzburg-Rallis model has non-trivial center, all of our integrations need to modulo the center. To simplify the notation, for any Lie algebra $\mathfrak{g}$ contained in $\mathfrak{gl}_n$ (in our case it will always be contained in $\mathfrak{gl}_6(F)$ or $\mathfrak{gl}_3(D)$), denote by $\mathfrak{g}_0$ the elements in $\mathfrak{g}$ whose trace (as an element in $\mathfrak{gl}_n$) is zero.

For a Levi subgroup M of G, let $\mathcal{P}(M)$ be the set of parabolic subgroups of G whose Levi part is M, $\mathcal{L}(M)$ be the set of Levi subgroups of G containing M, and $\mathcal{F}(M)$ be the set of parabolic subgroups of G containing M. We have a natural decomposition $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$, denote by $proj_M^G$ and $proj_G$ the projections of $\mathfrak{a}_M$ to each factors. The subspace $\mathfrak{a}_M^G$ has a set of coroots $\check{\Sigma}_M$, and for each $P \in \mathcal{P}(M)$, we can associate a positive chamber $\mathfrak{a}_P^+ \subset \mathfrak{a}_M$, a subset of simple coroots $\check{\Delta}_P \subset \check{\Sigma}_M$, and a subset of positive coroots $\check{\Sigma}_P \subset \check{\Sigma}_M$. For each $P = MU$, we can also define a function $H_P : G(F) \to \mathfrak{a}_M$ by $H_P(g) = H_M(m_g)$ where $g = m_g u_g k_g$ is the Iwasawa decomposition of g. According to Harish-Chandra, we can define the height function $\| \cdot \|$ on $G(F)$, taking values in $\mathbb{R}_{\geq 1}$, and a log-norm σ on $G(F)$ by $\sigma(g) = \sup(1, \log(\|g\|))$. Similarly, we can define the log-norm function on $\mathfrak{g}(F)$ as follows: fix a basis $\{X_i\}$ of $\mathfrak{g}(F)$ over F, for $X \in \mathfrak{g}(F)$, let $\sigma(X) = \sup(1, \sup\{-val_F(a_i)\})$, where a_i is the X_i-coordinate of X.

Let M_{min} be a minimal Levi subgroup of G, and let $A_{min} = A_{M_{min}}$. For each $P_{min} \in \mathcal{P}(M_{min})$, let $\Psi(A_{min}, P_{min})$ be the set of positive roots associated to P_{min}, and let $\Delta(A_{min}, P_{min}) \subset \Psi(A_{min}, P_{min})$ be the subset of simple roots.

For $x \in G$ (resp. $X \in \mathfrak{g}$), let $Z_G(x)$ (resp. $Z_G(X)$) be the centralizer of x (resp. X) in G, and let G_x (resp. G_X) be the neutral component of $Z_G(x)$ (resp. $Z_G(X)$). Accordingly, let $\mathfrak{g}_x$ (resp. $\mathfrak{g}_X$) be the Lie algebra of G_x (resp. G_X). For a function f on $G(F)$ (resp. $\mathfrak{g}(F)$), and $g \in G(F)$, let ${}^g f$ be the g-conjugation of f, i.e. ${}^g f(x) = f(g^{-1}xg)$ for $x \in G(F)$ (resp. ${}^g f(X) = f(g^{-1}Xg)$ for $X \in \mathfrak{g}(F)$).

Denote by $G_{ss}(F)$ the set of semisimple elements in $G(F)$, and by $G_{reg}(F)$ the set of regular elements in $G(F)$. The Lie algebra versions are denoted by $\mathfrak{g}_{ss}(F)$ and $\mathfrak{g}_{reg}(F)$, respectively. Now for $X \in G_{ss}(F)$, the operator $ad(x) - 1$ is defined and invertible on $\mathfrak{g}(F)/\mathfrak{g}_x(F)$. We define

$$D^G(x) = \mid \det((ad(x) - 1)_{|\mathfrak{g}(F)/\mathfrak{g}_x(F)}) \mid_F .$$

Similarly for $X \in \mathfrak{g}_{ss}(F)$, define

$$D^G(X) = \mid \det((ad(X))_{|\mathfrak{g}(F)/\mathfrak{g}_X(F)}) \mid_F .$$

For any subset $\Gamma \subset G(F)$, define $\Gamma^G := \{g^{-1}\gamma g \mid g \in G(F), \gamma \in \Gamma\}$. We say an invariant subset Ω of $G(F)$ is compact modulo conjugation if there exist a compact subset Γ such that $\Omega \subset \Gamma^G$. A G-domain on $G(F)$ (resp. $\mathfrak{g}(F)$) is an open subset of $G(F)$ (resp. $\mathfrak{g}(F)$) invariant under the $G(F)$-conjugation.

For two complex valued functions f and g on a set X with g taking values in the positive real numbers, we write that

$$f(x) \ll g(x)$$

and say that f is essentially bounded by g, if there exists a constant $c > 0$ such that for all $x \in X$, we have

$$|f(x)| \leq cg(x).$$

We say f and g are equivalent, which is denoted by

$$f(x) \sim g(x)$$

if f is essentially bounded by g and g is essentially bounded by f.

2.2 Measures

Through this paper, we fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$. If G is a connected reductive group, we may fix a non-degenerate symmetric bilinear form $< \cdot, \cdot >$ on $\mathfrak{g}(F)$ that is invariant under $G(F)$-conjugation. For any smooth compactly supported complex valued function $f \in C_c^\infty(\mathfrak{g}(F))$, we can define its Fourier transform $f \to \hat{f} \in C_c^\infty(\mathfrak{g}(F))$ to be

$$\hat{f}(X) = \int_{\mathfrak{g}(F)} f(Y)\psi(< X, Y >)dY \tag{2.1}$$

where dY is the selfdual Haar measure on $\mathfrak{g}(F)$ such that $\hat{\hat{f}}(X) = f(-X)$. Then we get a Haar measure on $G(F)$ such that the Jacobian of the exponential map equal to 1. If H is a subgroup of G such that the restriction of the bilinear form to $\mathfrak{h}(F)$ is also non-degenerate, then we can define the measures on $\mathfrak{h}(F)$ and $H(F)$ by the same method.

Let $Nil(\mathfrak{g})$ be the set of nilpotent orbits of $\mathfrak{g}$. For $\mathcal{O} \in Nil(\mathfrak{g})$ and $X \in \mathcal{O}$, the bilinear form $(Y, Z) \to < X, [Y, Z] >$ on $\mathfrak{g}(F)$ can be descented to a symplectic form on $\mathfrak{g}(F)/\mathfrak{g}_X(F)$. The nilpotent $\mathcal{O}$ has naturally a structure of F-analytic symplectic variety, which yields a selfdual measure on $\mathcal{O}$. This measure is invariant under the $G(F)$-conjugation.

If T is a subtorus of G such that the bilinear form is non-degenerate on $\mathfrak{t}(F)$, we can provide a measure on T by the method above, denoted by dt. On the other hand, we can define another measure $d_c t$ on $T(F)$ as follows: If T is split, we require the volume of the maximal compact subgroup of $T(F)$ is 1 under $d_c t$. In general, $d_c t$ is compatible with the measure $d_c t'$ defined on $A_T(F)$ and with the measure on $T(F)/A_T(F)$ of total volume 1. Then we have a constant number $\nu(T)$ such that $d_c t = \nu(T)dt$. In this paper, we will only use the measure dt, but in many cases we have to include the factor $\nu(T)$. Finally, if M is a Levi subgroup of G, we can define the Haar measure on $\mathfrak{a}_M^G$ such that the quotient

$$\mathfrak{a}_M^G/proj_M^G(H_M(A_M(F)))$$

is of volume 1.

2.3 The (G,M)-Family

From now on until Section 4, G will be a connected reductive group, and $\mathfrak{g}(F)$ be its Lie algebra, with a bilinear pairing invariant under conjugation. For a Levi subgroup M of G, we recall the notion of (G,M)-family introduced by Arthur. A (G,M)-family is a family $(c_P)_{P\in\mathcal{P}(M)}$ of smooth functions on $i\mathfrak{a}_M^*$ taking values in a locally convex topological vector space V such that for all adjacent parabolic subgroups $P, P' \in \mathcal{P}(M)$, the functions c_p and $c_{P'}$ coincide on the hyperplane supporting the wall that separates the positive chambers for P and P'. For such a (G,M)-family, one can associate an element $c_M \in V$ ([Ar81, Page 37]). If $L \in \mathcal{L}(M)$, for a given (G,M)-family, we can deduce a (G,L)-family. Denote by c_L the element in V associated to such (G,L)-family. If $Q = L_Q U_Q \in \mathcal{F}(L)$, we can deduce a (L_Q, L)-family from the given (G,M)-family, the element in V associated to which is denoted by c_L^Q.

If $(Y_P)_{P\in\mathcal{P}(M)}$ is a family of elements in $\mathfrak{a}_M$, we say it is a (G,M)-orthogonal set (resp. and positive) if the following condition holds: if P, P' are two adjacent elements of $\mathcal{P}(M)$, there exists a unique coroot $\check{\alpha}$ such that $\check{\alpha} \in \check{\Delta}_P$ and $-\check{\alpha} \in \check{\Delta}_{P'}$, we require that $Y_P - Y_{P'} \in \mathbb{R}\check{\alpha}$ (resp. $Y_P - Y_{P'} \in \mathbb{R}_{\geq 0}\check{\alpha}$). For $P \in \mathcal{P}(M)$, define a function c_P on $i\mathfrak{a}_M^*$ by $c_P(\lambda) = e^{-\lambda(Y_P)}$. Suppose that the family $(Y_P)_{P\in\mathcal{P}(M)}$ is a (G,M)-orthogonal set.

Then the family $(c_P)_{P\in\mathcal{P}(M)}$ is a (G,M)-family. If the family $(Y_P)_{P\in\mathcal{P}(M)}$ is positive, then the number c_M associated to this (G,M)-family is just the volume of the convex hull in $\mathfrak{a}_M^G$ generated by the set $\{Y_P \mid P\in\mathcal{P}(M)\}$. If $L\in\mathcal{L}(M)$, the (G,L)-family deduced from this (G,M)-family is the (G,L)-family associated to the (G,L)-orthogonal set $(Y_Q)_{Q\in\mathcal{P}(L)}$ where $Y_Q = proj_L(Y_P)$ for some $P\in\mathcal{P}(M)$ such that $P\subset Q$. It is easy to see that this is independent of the choice of P. Similarly, if $Q\in\mathcal{P}(L)$, then the (L,M)-family deduced from this (G,M)-family is the (L,M)-family associated to the (L,M)-orthogonal set $(Y_{P'})_{P'\in\mathcal{P}^L(M)}$ where $Y_{P'} = Y_P$ with P being the unique element of $\mathcal{P}(M)$ such that $P\subset Q$ and $P\cap L = P'$.

2.4 Weighted Orbital Integrals

If M is a Levi subgroup of G and K is a maximal open compact subgroup in good position with respect to M. For $g\in G(F)$, the family $(H_P(g))_{P\in\mathcal{P}(M)}$ is (G,M)-orthogonal and positive. Let $(v_P(g))_{P\in\mathcal{P}(M)}$ be the (G,M)-family associated to it and $v_M(g)$ be the number associated to this (G,M)-family. Then $v_M(g)$ is just the volume of the convex hull in $\mathfrak{a}_M^G$ generated by the set $\{H_P(g),\ P\in\mathcal{P}(M)\}$. The function $g\to v_M(g)$ is obviously left $M(F)$-invariant and right K-invariant.

If $f\in C_c^\infty(G(F))$ and $x\in M(F)\cap G_{reg}(F)$, define the weighted orbital integral to be

$$J_M(x,f) = D^G(x)^{1/2}\int_{G_x(F)\backslash G(F)} f(g^{-1}xg)v_M(g)dg. \tag{2.2}$$

Note the definition does depend on the choice of the hyperspecial open compact subgroup K. But we will see later that if f is strongly cuspidal, then this definition is independent of the choice of K.

Lemma 2.4.1. *With the notation as above, the following holds.*

1. *If $f\in C_c^\infty(G(F))$, the function $x\to J_M(x,f)$ defined on $M(F)\cap G_{reg}(F)$ is locally constant, invariant under $M(F)$-conjugation and has a compact support modulo conjugation.*

2. *There exists an integer $k\geq 0$, such that for every $f\in C_c^\infty(G(F))$, there exists $c>0$ such that*

$$|J_M(x,f)|\leq c(1+|\log D^G(x)|)^k$$

for every $x \in M(F) \cap G_{reg}(F)$.

Proof. See Lemma 2.3 of [W10]. □

The next result is due to Harish-Chandra (Lemma 4.2 of [Ar91]), which will be heavily used in Section 10 and Section 11. See [B15, Section 1.2] for a more general argument.

Proposition 2.4.2. *Let T be a torus of $G(F)$, and $\Gamma \subset G(F)$, $\Omega \subset T(F)$ be compact subsets. Then there exists $c > 0$ such that for every $x \in \Omega \cap G(F)_{reg}$ and $g \in G(F)$ with $g^{-1}xg \in \Gamma$, we have*

$$\sigma_T(g) \leq c(1+ \mid \log(D^G(x)) \mid) \tag{2.3}$$

where $\sigma_T(g) = \inf\{\sigma(tg) \mid t \in T(F)\}$.

2.5 Shalika Germs

For every $\mathcal{O} \in Nil(\mathfrak{g})$ and $f \in C_c^\infty(\mathfrak{g}(F))$, define the nilpotent orbital integral by

$$J_{\mathcal{O}}(f) = \int_{\mathcal{O}} f(X)dX.$$

Its Fourier transform is defined to be

$$\hat{J}_{\mathcal{O}}(f) = J_{\mathcal{O}}(\hat{f}).$$

For $\lambda \in F^\times$, define f^λ to be $f^\lambda(X) = f(\lambda X)$. Then it is easy to see that for $\lambda \in (F^\times)^2$, we have

$$J_{\mathcal{O}}(f^\lambda) =\mid \lambda \mid^{-dim(\mathcal{O})/2} J_{\mathcal{O}}(f). \tag{2.4}$$

Define $\delta(G) = \dim(G) - \dim(T)$, where T is any maximal torus of G (i.e. $\delta(G)$ is twice of the dimension of maximal unipotent subgroup if G split). There exists a unique function $\Gamma_{\mathcal{O}}$ on $\mathfrak{g}_{reg}(F)$, called the Shalika germ associated to $\mathcal{O}$, satisfies the following conditions:

$$\Gamma_{\mathcal{O}}(\lambda X) =\mid \lambda \mid_F^{(\delta(G)-\dim(\mathcal{O}))/2} \Gamma_{\mathcal{O}}(X) \tag{2.5}$$

for all $X \in \mathfrak{g}_{reg}(F), \lambda \in (F^\times)^2$, and for every $f \in C_c^\infty(\mathfrak{g}(F))$, there exists an neighborhood ω of 0 in $\mathfrak{g}(F)$ such that

$$J_G(X,f) = \Sigma_{\mathcal{O} \in Nil(\mathfrak{g})} \Gamma_{\mathcal{O}}(X) J_{\mathcal{O}}(f) \tag{2.6}$$

for every $X \in \omega \cap \mathfrak{g}_{reg}(F)$, where $J_G(X, f)$ is the orbital integral.

Harish-Chandra proved that there exists a unique function $\hat{j}$ on $\mathfrak{g}_{reg}(F) \times \mathfrak{g}_{reg}(F)$, which is locally constant on $\mathfrak{g}_{reg}(F) \times \mathfrak{g}_{reg}(F)$, and locally integrable on $\mathfrak{g}(F) \times \mathfrak{g}(F)$, such that for every $f \in C_c^\infty(\mathfrak{g}(F))$ and every $X \in \mathfrak{g}_{reg}(F)$,

$$J_G(X, \hat{f}) = \int_{\mathfrak{g}(F)} f(Y)\hat{j}(X, Y)dY. \tag{2.7}$$

Also, for all $\mathcal{O} \in Nil(\mathfrak{g})$, there exists a unique function $Y \to \hat{j}(\mathcal{O}, Y)$ on $\mathfrak{g}_{reg}(F)$, which is locally constant on $\mathfrak{g}_{reg}(F)$, and locally integrable on $\mathfrak{g}(F)$, such that for every $f \in C_c^\infty(\mathfrak{g}(F))$,

$$\hat{J}_{\mathcal{O}}(f) = \int_{\mathfrak{g}(F)} f(Y)\hat{j}(\mathcal{O}, Y)dY. \tag{2.8}$$

It follows that

$$\begin{aligned} \hat{j}(\lambda X, Y) &= |\lambda|_F^{\delta(G)/2}\, \hat{j}(X, \lambda Y), \\ \hat{j}(\mathcal{O}, \lambda Y) &= |\lambda|_F^{\dim(\mathcal{O})/2}\, \hat{j}(\mathcal{O}, Y) \end{aligned} \tag{2.9}$$

for all $X, Y \in \mathfrak{g}_{reg}(F), \mathcal{O} \in Nil(\mathfrak{g})$ and $\lambda \in (F^\times)^2$. Moreover, by the above discussion, if ω is an G-domain of $\mathfrak{g}(F)$ that is compact modulo conjugation and contains 0, there exists an G-domain ω' of $\mathfrak{g}(F)$ that is compact modulo conjugation and contains 0 such that for every $X \in \omega' \cap \mathfrak{g}_{reg}(F)$ and $Y \in \omega \cap \mathfrak{g}_{reg}(F)$,

$$\hat{j}(X, Y) = \Sigma_{\mathcal{O} \in Nil(\mathfrak{g})} \Gamma_{\mathcal{O}}(X)\hat{j}(\mathcal{O}, Y). \tag{2.10}$$

2.6 Induced Representations and the Intertwining Operators

Given a parabolic subgroup $P = MU$ of G and an admissible representation (τ, V_τ) of $M(F)$, let $(I_P^G(\tau), I_P^G(V_\tau))$ be the normalized parabolic induced representation: $I_P^G(V_\tau)$ consisting of smooth functions $e : G(F) \to V_\tau$ such that

$$e(mug) = \delta_P(m)^{1/2}\tau(m)e(g),\ m \in M(F),\ u \in U(F),\ g \in G(F).$$

And the $G(F)$ action is just the right translation.

For $\lambda \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$, let τ_λ be the unramified twist of τ (i.e. $\tau_\lambda(m) = \exp(\lambda(H_M(m)))\tau(m)$), and let $I_P^G(\tau_\lambda)$ be the induced representation. By the Iwasawa decomposition, every function $e \in I_P^G(\tau_\lambda)$ is determined by its restriction on K, and that space is invariant under the unramified twist. i.e. for any λ, we can realize the representation $I_P^G(\tau_\lambda)$ on the space $I_{K\cap P}^K(\tau_K)$ which consists of functions $e_K : K \to V_\tau$ such that

$$e(mug) = \delta_P(m)^{1/2}\tau(m)e(g),\ m \in M(F) \cap K,\ u \in U(F)\cap,\ g \in K.$$

Here τ_K is the restriction of the representation τ to the group K.

If τ is unitary, so is $I_P^G(\tau)$, the inner product on $I_P^G(V_\tau)$ can be realized as

$$(e, e') = \int_{P(F)\backslash G(F)} (e'(k), e(k))dk.$$

This is an invariant inner product under the representation $I_P^G(\tau_\lambda)$ for all $\lambda \in i\mathfrak{a}_M^*$.

Now we define the intertwining operator. For a Levi subgroup M of G, $P, P' \in \mathcal{P}(M)$, and $\lambda \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$, define the intertwining operator $J_{P'|P}(\tau_\lambda) : I_P^G(V_\tau) \to I_{P'}^G(V_\tau)$ to be

$$J_{P'|P}(\tau_\lambda)(e)(g) = \int_{(U(F)\cap U'(F))\backslash U'(F)} e(ug)du.$$

In general, the integral above is not absolutely convergent. But it is absolutely convergent for $Re(\lambda)$ sufficiently large, and it is $G(F)$-equivariant. By restricting to K, we can view $J_{P'|P}(\tau_\lambda)$ as a homomorphism from $I_{K\cap P}^K(V_{\tau_K})$ to $I_{K\cap P'}^K(V_{\tau_K})$. In general, $J_{P'|P}(\tau_\lambda)$ can be meromorphically continued to a function on $\mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}/i\mathfrak{a}_{M,F}^\vee$. Moreover, if we assume that τ is tempered, we have the following proposition which is due to Harish-Chandra.

Proposition 2.6.1. *With the notations above, assume that τ is tempered, then the intertwining operator $J_{P'|P}$ is absolutely convergent for all $\lambda \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$ with $< Re(\lambda), \check{\alpha} > > 0$ for every $\check{\alpha} \in \check{\Sigma}_P \cap \check{\Sigma}_(\bar{P}')$. Here $\Sigma(P)$ is the subsets of the roots of A_M that are positive with respect to P.*

We will use this proposition in Section 14 to show some generalized Jacquet integrals are absolutely convergent, and this integrals will occur in the open orbit method.

If τ is irreducible, by Schur's lemma, the operator $J_{P|\bar{P}}(\tau_\lambda)J_{\bar{P}|P}(\tau_\lambda)$ is a scalar for generic λ, let $j(\tau_\lambda)$ be the scalar, this is independent of the choice of P. We can normalize the intertwining operator by a complex valued function $r_{P'|P}(\tau_\lambda)$ such that the normalized intertwining operator

$$R_{P'|P}(\tau_\lambda) = r_{P'|P}(\tau_\lambda)^{-1} J_{P'|P}(\tau_\lambda)$$

satisfies the conditions of Theorem 2.1 of [Ar89]. The key conditions are

1. For $P, P', P'' \in \mathcal{P}(M)$, $R_{P''|P'}(\tau_\lambda) R_{P'|P}(\tau_\lambda) = R_{P''|P}(\tau_\lambda)$.

2. Suppose τ is tempered, for $\lambda \in i\mathfrak{a}^*_{M,F}$, $R_{P'|P}(\tau_\lambda)$ is holomorphic and unitary.

3. The normalized intertwining operator are compatible with the unramified twist and the parabolic induction.

2.7 Weighted Characters

Let M be a Levi subgroup, and τ be a tempered representation of $M(F)$. For $P, P' \in \mathcal{P}(M)$, we have defined the normalized intertwining operator $R_{P'|P}(\tau_\lambda)$ for $\lambda \in i\mathfrak{a}^*_M$. Fix P, for every $P' \in \mathcal{P}(M)$, define the function $\mathcal{R}_{P'}(\tau)$ on $i\mathfrak{a}^*_M$ by

$$\mathcal{R}_{P'}(\tau, \lambda) = R_{P'|P}(\tau)^{-1} R_{P'|P}(\tau_\lambda).$$

This function takes value in the space of endomorphisms of $I^K_{P\cap K}(\tau_K)$ (not necessarily commutes with the G-action). Recall that this space is invariant under the unramified twist. By [Ar81], this is a (G, M)-family. Then for $L \in \mathcal{L}(M)$ and $Q \in \mathcal{F}(L)$, we can associate an operator $\mathcal{R}^Q_L(\tau)$ to this (G, M) family. We define the weighted character of τ to be the distribution $f \to J^Q_L(\tau, f)$ given by $J^Q_L(\tau, f) = \mathrm{tr}(\mathcal{R}^Q_L(\tau) I^G_P(\tau)(f))$ for every $f \in C^\infty_c(G(F))$. This is independent of the choice of P but depends on K and the way we normalized the intertwining operators. In particular, if $L = Q = G$, the distribution $J^G_G(\tau, f)$ is just θ_π for $\pi = I^G_P(\tau)$ where $\theta_\pi(f) = tr(\pi(f))$.

2.8 The Harish-Chandra-Schwartz Space

Let P_{min} be a minimal parabolic subgroup of G, and let K be a maximal open compact subgroup in good position with respect to M. Then we have the Iwasawa decomposition

$G(F) = P_{min}(F)K$. Consider the normalized induced representation

$$I^G_{P_{min}}(1) := \{e \in C^\infty(G(F)) \mid e(pg) = \delta_{P_{min}}(p)^{1/2}e(g) \text{ for all } p \in P_{min}(F), g \in G(F)\}$$

and we equip the representation with the inner product

$$(e, e') = \int_K e(k)\bar{e'}(k)dk.$$

Let $e_K \in I^G_{P_{min}}(1)$ be the unique function such that $e_K(k) = 1$ for all $k \in K$.

Definition 2.8.1. *The Harish-Chandra function Ξ^G is defined to be*

$$\Xi^G(g) = (I^G_{P_{min}}(1)(g)e_K, e_K).$$

Remark 2.8.2. *The function Ξ^G depends on the various choices we made, but this doesn't matter since different choices give us equivalent functions and the function Ξ^G will only be used in estimations.*

The next proposition summarize some basic properties of the function Ξ^G, the proof of the proposition can be found in [W03]. Also see Proposition 1.5.1 of [B15].

Proposition 2.8.3.

1. *Let*

   $$M^+_{min} = \{m \in M_{min}(F) \mid\mid \alpha(m) \mid\leq 1 \text{ for all } \alpha \in \Psi(A_{M_{min}}, P_{min})\}.$$

   Then there exists $d > 0$ such that

   $$\delta_{P_{min}}(m)^{1/2} \ll \Xi^G(m) \ll \delta_{P_{min}}(m)^{1/2}\sigma_0(m)^d$$

   for all $m \in M^+_{min}$.

2. *There exists $d > 0$ such that $\Xi^G(g) \ll \delta_{P_{min}}(m_{P_{min}}(g))^{1/2}\sigma_0(g)^d$ for all $g \in G(F)$, here $m_{P_{min}}(g)$ is the M_{min}-part of g under the Iwasawa decomposition $G = M_{min}U_{min}K$.*

3. *Let $P = MU$ be a parabolic subgroup containing P_{min}, then we have*

   $$\Xi^G(g) = \int_K \delta_P(m_P(kg))^{1/2}\Xi^M(m_P(kg))dk$$

   for all $g \in G(F)$, here $m_P(g)$ is the M-part of g under the Iwasawa decomposition $G = MUK$.

4. *Let $P = MU$ be a parabolic subgroup of G. Then for all $d > 0$, there exist $d' > 0$ such that*

$$\delta_P(m)^{1/2} \int_{U(F)} \Xi^G(mu)\sigma_0(mu)^{-d'} du \ll \Xi^M(m)\sigma_0(m)^{-d}$$

for all $m \in M(F)$.

5. *There exists $d > 0$ such that $\int_{G(F)} \Xi^G(g)^2\sigma(g)^{-d}dg$ is convergent.*

6. *We have the equality*

$$\int_K \Xi^G(g_1kg_2)dk = \Xi^G(g_1)\Xi^G(g_2)$$

for all $g_1, g_2 \in G(F)$.

For $f \in C^\infty(G(F))$ and $d \in \mathbb{R}$, let

$$p_d(f) = \sup_{g \in G(F)} \{|f(g)|\Xi^G(g)^{-1}\sigma(g)^d\}.$$

If F is p-adic, we define the Harish-Chandra-Schwartz space to be

$$\mathcal{C}(G(F)) = \{f \in C^\infty(G(F)) | p_d(f) < \infty, \forall d > 0\}.$$

If $F = \mathbb{R}$, for $u, v \in \mathcal{U}(\mathfrak{g})$ and $d \in \mathbb{R}$, let

$$p_{u,v,d}(f) = p_d(R(u)L(v)f)$$

where "R" stands for the right translation, "L" stands for the left translation and $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra. We define the Harish-Chandra-Schwartz space to be

$$\mathcal{C}(G(F)) = \{f \in C^\infty(G(F)) | p_{u,v,d}(f) < \infty, \forall d > 0,\ u, v \in \mathcal{U}(\mathfrak{g})\}.$$

We also need the weak Harish-Chandra-Schwartz space $\mathcal{C}^w(G(F))$. For $d > 0$, let

$$\mathcal{C}_d^w(G(F)) = \{f \in C^\infty(G(F)) | p_{-d}(f) < \infty\}$$

if F is p-adic. And let

$$\mathcal{C}_d^w(G(F)) = \{f \in C^\infty(G(F)) | p_{u,v,-d}(f) < \infty, \forall u, v \in \mathcal{U}(\mathfrak{g})\}$$

if $F=\mathbb{R}$. Then the weak Harish-Chandra-Schwartz space is defined to be

$$\mathcal{C}^w(G(F)) = \cup_{d>0}\mathcal{C}_d^w(G(F)).$$

Also we can define the Harish-Chandra-Schwartz space (resp. weak Harish-Chandra-Schwartz space) with given unitary central character χ: let $\mathcal{C}(G(F),\chi)$ (resp. $\mathcal{C}^w(G(F),\chi)$) be the Mellin transform of the space $\mathcal{C}(G(F))$ (resp. $\mathcal{C}^w(G(F))$) with respect to χ.

2.9 The Harish-Chandra-Plancherel Formula

Since the Ginzburg-Rallis model has nontrivial center, we only introduce the Plancherel formula with given central character. We fix an unitary character χ of $Z_G(F)$. For every $M\in\mathcal{L}(M_{min})$, fix an element $P\in\mathcal{P}(M)$. Let $\Pi_2(M,\chi)$ be the set of discrete series of $M(F)$ whose central character agree with χ on $Z_G(F)$. Then $i\mathfrak{a}_{M,0}^*$ acts on $\Pi_2(M,\chi)$ by the unramified twist. Let $\{\Pi_2(M,\chi)\}$ be the set of orbits under this action. For every orbit $\mathcal{O}$, and for a fixed $\tau\in\mathcal{O}$, let $i\mathfrak{a}_{\mathcal{O}}^\vee$ be the set of $\lambda\in i\mathfrak{a}_{M,0}^*$ such that the representation τ and τ_λ are equivalent, which is a finite set. For $\lambda\in i\mathfrak{a}_{M,0}^*$, define the Plancherel measure to be

$$\mu(\tau_\lambda) = j(\tau_\lambda)^{-1}d(\tau)$$

where $d(\tau)$ is the formal degree of τ, which is invariant under the unramified twist, and $j(\tau_\lambda)$ is defined in Section 2.6. Then for $f\in\mathcal{C}(G(F),\chi^{-1})$, the Harish-Chandra-Plancherel formula is

$$\begin{aligned} f(g) &= \Sigma_{M\in\mathcal{L}(M_{min})}|W^M||W^G|^{-1}\Sigma_{\mathcal{O}\in\{\Pi_2(M,\chi)\}}|i\mathfrak{a}_{\mathcal{O}}^\vee|^{-1} \\ &\quad \int_{i\mathfrak{a}_{M,0}^*}\mu(\tau_\lambda)\mathrm{tr}(I_P^G(\tau_\lambda)(g^{-1})I_P^G(\tau_\lambda)(f))d\lambda. \end{aligned}$$

The proof of the above formula can be found in [W03] for the p-adic case, and in [Ar75] for the real case.

To simplify our notation, let $\Pi_{temp}(G,\chi)$ be the union of $I_P^G(\tau)$ for $P=MN$, $M\in\mathcal{L}(M_{min})$, $\tau\in\mathcal{O}$ and $\mathcal{O}\in\{\Pi_2(M,\chi)\}$. We define a Borel measure $d\pi$ on $\Pi_{temp}(G,\chi)$ such that

$$\int_{\Pi_{temp}(G,\chi)}\varphi(\pi)d\pi = \Sigma_{M\in\mathcal{L}(M_{min})}|W^M||W^G|^{-1}\Sigma_{\mathcal{O}\in\{\Pi_2(M,\chi)\}}|i\mathfrak{a}_{\mathcal{O}}^\vee|^{-1}\int_{i\mathfrak{a}_{M,0}^*}\varphi(I_P^G(\tau_\lambda))d\lambda$$

for every compact supported function φ on $\Pi_{temp}(G,\chi)$. Here by saying a function φ is compactly supported on $\Pi_{temp}(G,\chi)$ we mean that it is supported on finitely many orbit $\mathcal{O}$ and for every such orbit $\mathcal{O}$, it is compactly supported. Note that the second condition is automatic if F is p-adic. Then the Harish-Chandra-Plancherel formula above becomes

$$f(g) = \int_{\Pi_{temp}(G,\chi)} \mathrm{tr}(\pi(g^{-1})\pi(f))\mu(\pi)d\pi.$$

We also need the matrical Paley-Wiener Theorem. Let $C^{\infty}(\Pi_{temp}(G,\chi))$ be the space of functions $\pi \in \Pi_{temp}(G,\chi) \to T_{\pi} \in \mathrm{End}(\pi)^{\infty}$ such that it is smooth on every orbits $\mathcal{O}$ as functions from $\mathcal{O}$ to $\mathrm{End}(\pi)^{\infty} \simeq \mathrm{End}(\pi_K)^{\infty}$. Now we define $\mathcal{C}(\Pi_{temp}(G,\chi))$ to be a subspace of $C^{\infty}(\Pi_{temp}(G,\chi))$ consisting of those $T : \pi \to T_{\pi}$ such that

1. If F is p-adic, T is nonzero on finitely many orbits $\mathcal{O}$.
2. If $F = \mathbb{R}$, for all parabolic subgroup $P = MU$ and for all differential operator with constant coefficients D on $i\mathfrak{a}_M^*$, the function $DT : \sigma \in \Pi_2(M,\chi) \to D(\lambda \to T_{I_P^G(\sigma_\lambda)})$ satisfies $p_{D,u,v,k}(T) = \sup_{\sigma\in\Pi_2(M,\chi)} ||DT(\sigma)||_{u,v} N(\sigma)^k < \infty$ for all $u, v \in \mathcal{U}(\mathfrak{k})$ and $k \in \mathbb{N}$. Here $||DT(\sigma)||_{u,v}$ is the norm of the operator $\sigma(u)DT(\sigma)\sigma(v)$ and $N(\sigma)$ is the norm on the set of all tempered representations (See Section 2.2 of [B15]).

Then the matrical Paley-Wiener Theorem states that we have an isomorphism between $\mathcal{C}(G,\chi^{-1})$ and $\mathcal{C}(\Pi_{temp}(G,\chi))$ given by

$$f \in \mathcal{C}(G,\chi^{-1}) \to (\pi \in \Pi_{temp}(G,\chi) \to \pi(f) \in \mathrm{End}(\pi)^{\infty})$$

and

$$T \in \mathcal{C}(\Pi_{temp}(G,\chi)) \to f_T(g) = \int_{\Pi_{temp}(G,\chi)} \mathrm{tr}(\pi(g^{-1})T_{\pi})\mu(\pi)d\pi.$$

Chapter 3

Strongly Cuspidal Functions and Quasi-Characters

In this chapter, we will study the strongly cuspidal functions and quasi-characters. These are the main ingredients of our trace formula. In Section 3.1, we consider the neighborhood of semisimple elements. In Section 3.2, we will define quasi-characters both on the group level and on the Lie algebra level. In Section 3.3, we study the behavior of quasi-characters under the parabolic induction. This will be used in the spectral side of the trace formula when we are trying to reduce our problems to the discrete series. In Section 3.4, we will define the strongly cuspidal functions and talk about some geometric properties of them. This will serve as our test functions in the trace formula. Moreover, for each strongly cuspidal function f, we will define a quasi-character θ_f. This distribution will appear on both sides of the trace formula. In Section 3.5, we will establish some spectral properties of the strongly cuspidal functions.

After that, we will talk about the localization of various objects. This will be used in the geometric side of the trace formula when we try to reduce the problems to the Lie algebra case. In Section 3.6, we study the localization of general quasi-characters. Then in Section 3.7, we will talk about the localization of θ_f. Finally, in Section 3.8, we will talk about the pseudo coefficients of the discrete series, which will be used in Section 13 when we are trying to deduce the multiplicity formula from the trace formula.

3.1 Neighborhoods of Semisimple Elements

Definition 3.1.1. *For every $x \in G_{ss}(F)$, we say a subset $\omega \subset \mathfrak{g}_x(F)$ is a good neighborhood of 0 if it satisfies the following seven conditions, together with condition $(7)_\rho$ for finitely many finite dimensional algebraic representations (ρ, V) of G that will be fixed in advance ([W10, Section 3.1]):*

(1) ω is an G_x-domain, compact modulo conjugation, invariant under $Z_G(x)(F)$ conjugation and contains 0.

(2) The exponential map is defined on ω, i.e. it is a homeomorphism between ω and $\exp(\omega)$, and is G_x-equivariant, where the action is just conjugation.

(3) For every $\lambda \in F^\times$ with $|\lambda| \leq 1$, we have $\lambda\omega \subset \omega$.

(4) We have

$$\{g \in G(F) \mid g^{-1}x\exp(\omega)g \cap x\exp(\omega) \neq \emptyset\} = Z_G(x)(F). \tag{3.1}$$

(5) For every compact subset $\Gamma \subset G(F)$, there exists a compact subset $\Gamma' \subset G(F)$ such that

$$\{g \in G(F) \mid g^{-1}x\exp(\omega)g \cap \Gamma = \emptyset\} \subset G_x(F)\Gamma'.$$

(6) Fix a real number $c_F > 0$ such that $c_F^k < |(k+1)!|_F$ for every integer $k \geq 1$. Then for every maximal subtorus $T \subset G_x$, every algebraic character χ of T and every element $X \in \mathfrak{t}(F) \cap \omega$, we have $|\chi(X)|_F < c_F$.

(7) Consider an eigenspace $W \subset \mathfrak{g}(F)$ for the operator $ad(x)$, and let λ be the eigenvalue. If $X \in \omega$, then $ad(X)$ preserve W. Let W_X be an eigenspace of it with eigenvalue μ. Then it is easy to see that W_X is also an eigenspace for the operator $ad(x\exp(X))$, with eigenvalue $\lambda\exp(\mu)$. Now suppose $\lambda \neq 1$. Then

$$|\lambda\exp(\mu) - 1|_F = |\lambda - 1|_F .$$

$(7)_\rho$ If we fix a finite dimensional algebraic representation (ρ, V) of G, by replacing the adjoint representation by (ρ, V) in (7), we can define condition $(7)_\rho$ in a similar way.

The properties for good neighborhoods are summarized below, the details of which will be referred to [W10, Section 3].

Proposition 3.1.2. *The following hold.*

1. *If* ω_0 *is a neighborhood of* 0 *in* $\mathfrak{g}_x(F)$*, there exists a good neighborhood* ω *of 0 such that* $\omega \subset \omega_0^{G_x}$*.*

2. $\Omega = (x\exp(\omega))^G$ *is an G-domain in* $G(F)$*, and has compactly support modulo conjugation.*

3. *For every* $X \in \omega$*,* $Z_G(x\exp(X))(F) \subset Z_G(x)(F)$ *and* $G_{x\exp(X)} = (G_x)_X \subset G_x$*.*

4. *The exponential map between* ω *and* $\exp(\omega)$ *preserve measures, i.e. the Jacobian of the map equals* 1*.*

5. *For every* $X \in \omega$*,* $D^G(x\exp(X)) = D^G(x)D^{G_x}(X)$*.*

Proof. See Section 3.1 of [W10]. □

3.2 Quasi-Characters of $G(F)$ and $\mathfrak{g}(F)$

If θ is a smooth function defined on $G_{reg}(F)$, invariant under $G(F)-$conjugation. We say it is a quasi-character on $G(F)$ if, for every $x \in G_{ss}(F)$, there is a good neighborhood ω_x of 0 in $\mathfrak{g}_x(F)$, and for every $\mathcal{O} \in Nil(\mathfrak{g}_x)$, there exists coefficient $c_{\theta,\mathcal{O}}(x) \in \mathbb{C}$ such that

$$\theta(x\exp(X)) = \Sigma_{\mathcal{O}\in Nil(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x)\hat{j}(\mathcal{O}, X) \tag{3.2}$$

for every $X \in \omega_{x,reg}$. It is easy to see that $c_{\theta,\mathcal{O}}(x)$ are uniquely determined by θ. If θ is a quasi-character on $G(F)$ and $\Omega \subset G(F)$ is an open G-domain, then $\theta 1_\Omega$ is still a quasi-character.

For the Lie algebra case, let θ be a function on $\mathfrak{g}_{reg}(F)$, invariant under $G(F)-$conjugation. We say it is a quasi-character on $\mathfrak{g}(F)$ if for every $X \in \mathfrak{g}_{ss}(F)$, there exists an open G_X-domain ω_X in $\mathfrak{g}_X(F)$, containing 0, and for every $\mathcal{O} \in Nil(\mathfrak{g}_X)$, there exists $c_{\theta,\mathcal{O}}(X) \in \mathbb{C}$ such that

$$\theta(X+Y) = \Sigma_{\mathcal{O}\in Nil(\mathfrak{g}_X)} c_{\theta,\mathcal{O}}(X)\hat{j}(\mathcal{O}, Y) \tag{3.3}$$

for every $Y \in \omega_{X,reg}$. If θ is a quasi-character on $\mathfrak{g}(F)$, define $c_{\theta,\mathcal{O}} = c_{\theta,\mathcal{O}}(0)$. If $\lambda \in F^{\times}$, then $\theta^{\lambda}(X) = \theta(\lambda X)$ is still a quasi-character on $\mathfrak{g}(F)$. By Section 4.2 of [W10], for every $\mathcal{O} \in Nil(\mathfrak{g}_X)$, we have

$$c_{\theta^{\lambda},\mathcal{O}}(\lambda^{-1}X) = \mid \lambda \mid_F^{-dim(\mathcal{O})/2} c_{\theta,\mathcal{O}}(X). \tag{3.4}$$

3.3 Quasi-Characters Under Parabolic Induction

Let M be a Levi subgroup of G. Given an invariant distribution D^M on $M(F)$, we define the induced distribution $D = I_M^G(D^M)$ on $G(F)$ as follows.

Fix a parabolic subgroup $P = MU \in \mathcal{P}(M)$ and a hyperspecial maximal compact subgroup K. Assume that the Haar measure on $G(F)$, $M(F)$, $U(F)$ and K are compatible, i.e. $\int_G = \int_M \int_U \int_K$. For $f \in C_c^{\infty}(G(F))$, define $f_P \in C_c^{\infty}(M(F))$ to be

$$f_P(m) = \delta_P(m)^{1/2} \int_K \int_{U(F)} f(k^{-1}muk)dudk.$$

Then we define $D(f) = D^M(f_P)$.

If D^M is represented by a function θ^M on $M_{reg}(F)$, locally integrable on $M(F)$ and invariant under conjugation, i.e. $D^M(f) = \int_{M(F)} f(m)\theta^M(m)dm$ for all $f \in C_c^{\infty}(M(F))$. Then D is also represented by a function θ on $G_{reg}(F)$ defined by

$$\theta(x) = \Sigma_{x' \in \mathcal{X}^M(x)} D^G(x)^{-1/2} D^M(x')^{1/2} \theta^M(x'),\ x \in G_{reg}(F).$$

Here $\mathcal{X}^M(x)$ is the set of the $M(F)$-conjugation classes in the $G(F)$-conjugation class of x. In particular, if τ is an irreducible admissible representation of $M(F)$ and $\pi = I_P^G(\tau)$, then $\theta_{\pi} = I_M^G(\theta_{\tau})$.

Now we talk about the parabolic induction of quasi-characters. If $\mathcal{O}^M \in Nil(\mathfrak{m})$ and $\mathcal{O} \in Nil(\mathfrak{g})$, we say $\mathcal{O}$ is contained in the induced orbit of $\mathcal{O}^M$ if the intersection $\mathcal{O} \cap (\mathcal{O}^M + \mathfrak{u}(F))$ is a nonempty open subset in $\mathcal{O}^M + \mathfrak{u}(F)$. The following result is Lemma 2.3 of [W12].

Lemma 3.3.1. *If θ^M is a quasi-character of $M(F)$ and $\theta = I_M^G(\theta^M)$, then the followings hold.*

1. *θ is a quasi-character of $G(F)$.*

2. *If $x \in G_{ss}(F)$ and $\mathcal{O} \in Nil(\mathfrak{g}_x)$ is a regular orbit, then we have*

$$\begin{aligned} c_{\theta,\mathcal{O}}(x) &= \Sigma_{x' \in \mathcal{X}^M(x)} \Sigma_{g \in \Gamma_{x'}/G_x(F)} \Sigma_{\mathcal{O}'} D^G(x)^{-1/2} D^M(x')^{1/2} \\ & [Z_M(x')(F) : M_{x'}(F)]^{-1} c_{\theta^M,\mathcal{O}'}(x'). \end{aligned}$$

Here $\mathcal{O}'$ runs over elements in $Nil(\mathfrak{m}_{x'})$ such that $g\mathcal{O}$ is contained in the induced orbit of $\mathcal{O}'$. And for $x' \in \mathcal{X}^M(x)$, $\Gamma_{x'}$ is the set of $g \in G(F)$ such that $gxg^{-1} = x'$.

3.4 Strongly Cuspidal Functions

If $f \in \mathcal{C}(Z_G(F)\backslash G(F))$, we say f is strongly cuspidal if for every proper parabolic subgroup $P = MU$ of G, and for every $x \in M(F)$, we have

$$\int_{U(F)} f(xu)du = 0. \tag{3.5}$$

The most basic example of strongly cuspidal functions is given by the matrix coefficients of a supercuspidal representation.

The following proposition is easy to prove, following mostly from the definition. See Section 5.1 of [W10].

Proposition 3.4.1. *The following hold.*

1. *f is strongly cuspidal if and only if for every proper parabolic subgroup $P = MU$ of G, and for every $x \in M(F)$, we have*

$$\int_{U(F)} f(u^{-1}xu)du = 0. \tag{3.6}$$

2. *If Ω is a G-domain in $G(F)$ and if f is strongly cuspidal, then $f 1_\Omega$ is strongly cuspidal.*

3. *If f is strongly cuspidal, so is ${}^g f$ for every $g \in G(F)$.*

Now we study the weighted orbital integral associated to strongly cuspidal functions. The following lemma is proved in Section 5.2 of [W10].

Lemma 3.4.2. *Let M be a Levi subgroup of G and K be a hyperspecial open compact subgroup with respect to M. If $f \in \mathcal{C}(Z_G(F)\backslash G(F))$ is strongly cuspidal and $x \in M(F) \cap G_{reg}(F)$, then the following hold.*

1. *The weighted orbital integral $J_M(x,f)$ does not depend on the choice of K.*

2. *For every $y \in G(F)$, we have $J_M(x, {}^y f) = J_M(x,f)$.*

3. *If $A_{G_x} \neq A_M$, then $J_M(x,f) = 0$.*

For $x \in G_{reg}(F)$, let $M(x)$ be the centralizer of A_{G_x} in G, which is clearly a Levi subgroup of G. For any strongly cuspidal f belonging to the space $\mathcal{C}(Z_G(F)\backslash G(F))$, define the function θ_f on $Z_G(F)\backslash G_{reg}(F)$ by

$$\theta_f(x) = (-1)^{a_{M(x)}-a_G}\nu(G_x)^{-1}D^G(x)^{-1/2}J_{M(x)}(x,f). \tag{3.7}$$

Here a_G is the dimension of A_G, and the same for $a_{M(x)}$. By the lemma above, the weighted orbital integral is independent of the choice of the hyperspecial open compact subgroup, and so is the function θ_f.

Proposition 3.4.3. *The following hold.*

1. *The function θ_f is invariant under $G(F)$-conjugation, has a compact support modulo conjugation and modulo the center, is locally integrable on $Z_G(F)\backslash G(F)$ and locally constant on $Z_G(F)\backslash G_{reg}(F)$.*

2. *θ_f is a quasi-character.*

Proof. The first part is Lemma 5.3 of [W10], the second part is Corollary 5.9 of the loc. cit. □

The function θ_f will show up on both sides of the trace formula. Here we only write down the results for the trivial central character case, but the argument can be easily extended to the non-trivial central character case (i.e. $f \in \mathcal{C}(Z_G(F)\backslash G(F),\chi)$), or the case without central character (i.e. $f \in \mathcal{C}(G(F))$).

Similarly, we can define strongly cuspidal functions on the Lie algebra.

Definition 3.4.4. *We say a function $f \in C_c^\infty(\mathfrak{g}_0(F))$ is strongly cuspidal if for every proper parabolic subgroup $P = MU$, and for every $X \in \mathfrak{m}(F)$, we have*

$$\int_{\mathfrak{u}(F)} f(X+Y)dY = 0.$$

This is equivalent to say that for every proper parabolic subgroup $P = MU$, and for every $X \in \mathfrak{m}(F)$, we have

$$\int_{U(F)} f(u^{-1}Xu)du = 0.$$

If $f \in C_c^\infty(\mathfrak{g}_0(F))$ is strongly cuspidal, we define a function θ_f on $\mathfrak{g}_{0,reg}(F)$ by

$$\theta_f(X) = (-1)^{a_{M(X)}-a_G}\nu(G_X)^{-1}D^G(X)^{-1/2}J_{M(X)}(X,f). \tag{3.8}$$

Here $M(X)$ is the centralizer of A_{G_X} in G, a_G is the dimension of A_G, and the same for $a_{M(X)}$. We have a similar result as Proposition 3.4.3.

Proposition 3.4.5. *If $f \in C_c^\infty(\mathfrak{g}_0(F))$ is strongly cuspidal, θ_f is independent of the choice of K. (Recall we need to fix the open compact subgroup K in the definition of orbital integral.) And in this case, θ_f is a quasi-character.*

3.5 Some Spectral Properties of the Strongly Cuspidal Functions

We first study the weighted characters associated to the strongly cuspidal functions.

Lemma 3.5.1. *If $f \in \mathcal{C}(Z_G(F)\backslash G(F), \chi^{-1})$ is strongly cuspidal, M is a Levi subgroup of G and τ is a tempered representation of $M(F)$ whose central character equals χ on $Z_G(F)$, then the following hold.*

1. *For any $L \in \mathcal{L}(M)$ and $Q \in \mathcal{F}(L)$, $J_L^Q(\tau, f) = 0$ if $L \neq M$ or $Q \neq G$.*
2. *If τ is induced from a proper parabolic subgroup of M, then $J_M^G(\tau, f) = 0$.*
3. *For $x \in G(F)$, we have $J_{xMx^{-1}}^G(x\tau x^{-1}, f) = J_M^G(\tau, f)$.*
4. *The weight character $J_M^G(\tau, f)$ does not depend on the choice of K, and the way we normalize the intertwining operators.*

Proof. See Section 2.2 of [W12], or Section 5.4 of [B15]. □

Now we talk about the spectral characterization of the strongly cuspidal functions. The following result is a direct consequence of the matrical Paley-Wiener Theorem in Section 2.9.

Proposition 3.5.2. *For $f \in \mathcal{C}(Z_G(F)\backslash G(F), \chi^{-1})$, the following are equivalent.*

1. *f is strongly cuspidal.*
2. *For any proper parabolic subgroup $P = MU$, and for any tempered representation τ of $M(F)$ whose central character equals χ on $Z(F)$, we have $\mathrm{tr}(\pi(f)) = 0$ for $\pi = I_P^G(\tau)$.*

For the rest of this section, we assume that G is $GL_n(D)$ for some division algebra D/F and $n \geq 1$. In particular, all irreducible tempered representation π of $G(F)$ is of the form $\pi = I_M^G(\tau)$ for some $\tau \in \Pi_2(M)$. For such π, let χ be the central character of π. For $f \in \mathcal{C}(Z_G(F)\backslash G(F), \chi^{-1})$ strongly cuspidal, define

$$\theta_f(\pi) = (-1)^{a_G - a_M} J_M^G(\tau, f). \tag{3.9}$$

Proposition 3.5.3. *For every $f \in \mathcal{C}(Z_G(F)\backslash G(F), \chi^{-1})$ strongly cuspidal, we have*

$$\theta_f = \int_{\Pi_{temp}(G,\chi)} \theta_f(\pi)\bar{\theta}_\pi d\pi.$$

Proof. This is just Proposition 5.6.1 of [B15]. The only thing worth to mention is that the function $D(\pi)$ in the loc. cit. is identically 1 in our case since we assume $G = GL_n(D)$. $\square$

To end this section, we need a local trace formula for strongly cuspidal functions. It will be used in Chapter 8 for the proof of the spectral side of our trace formula. For $f \in \mathcal{C}(G(F), \chi^{-1})$, $f' \in \mathcal{C}(G(F), \chi)$ and $g_1, g_2 \in G(F)$, set

$$K_{f,f'}^A(g_1, g_2) = \int_{Z_G(F)\backslash G(F)} f(g_1^{-1} g g_2) f'(g) dg.$$

By Proposition 2.8.3, the integral above is absolutely convergent.

Theorem 3.5.4. *1. For all $d \geq 0$, there exist $d' \geq 0$, a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}(G(F), \chi^{-1})$ and a continuous semi-norm $\nu'_{d,d'}$ on $\mathcal{C}(G(F), \chi)$ such that*

$$|K_{f,f'}^A(g_1, g_2)| \leq \nu_{d,d'}(f)\nu'_{d,d'}(f')\Xi^G(g_1)\sigma_0(g_1)^{-d}\Xi^G(g_2)\sigma_0(g_2)^{d'}$$

and

$$|K_{f,f'}^A(g_1, g_2)| \leq \nu_{d,d'}(f)\nu'_{d,d'}(f')\Xi^G(g_1)\sigma_0(g_1)^{d'}\Xi^G(g_2)\sigma_0(g_2)^{-d}.$$

2. *Assume that f is strongly cuspidal for the rest part of the Theorem. Then for all $d \geq 0$, there exist a continuous semi-norm ν_d on $\mathcal{C}(G(F), \chi^{-1})$ and a continuous semi-norm ν_d' on $\mathcal{C}(G(F), \chi)$ such that $|K_{f,f'}^A(g,g)| \leq \nu_d(f)\nu_d'(f')\Xi^G(g)^2\sigma_0(g)^{-d}$.*

3. *There exists $c > 0$ such that for all $d \geq 0$, and there exists $d' \geq 0$ such that $|K_{f,f'}^A(g,hg)| \ll \Xi^G(g)^2\sigma_0(g)^{-d}e^{c\sigma_0(h)}\sigma_0(h)^{d'}$.*

4. *Set $J^A(f,f') = \int_{Z_G(F)\backslash G(F)} K_{f,f'}^A(g,g)dg$. This is absolutely convergent by part (2). Then we have*

$$J^A(f,f') = \int_{Temp(G,\chi)} \theta_f(\pi)\theta_{\bar{\pi}}(f')d\pi.$$

Proof. This is just Theorem 5.5.1 of [B15]. □

3.6 The Localization of Quasi-Characters

We fix $x \in G_{ss}(F)$ and a good neighborhood ω of 0 in $\mathfrak{g}_x(F)$. If θ is a quasi-character of $G(F)$, we define a function $\theta_{x,\omega}$ on ω by

$$\theta_{x,\omega}(X) = \begin{cases} \theta(x\exp(X)), & \text{if } X \in \omega; \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Then $\theta_{x,\omega}$ is a quasi-character of $\mathfrak{g}_x(F)$, and we have $c_{\theta,\mathcal{O}}(x\exp(X)) = c_{\theta_{x,\omega},\mathcal{O}}(X)$ for every $X \in \omega \cap \mathfrak{g}_{x,ss}(F)$ and $\mathcal{O} \in Nil(\mathfrak{g}_{x,X})$ (Note we have $G_{x\exp(X)} = (G_x)_X$ since ω is a good neighborhood). In particular, by taking $X = 0$ we have $c_{\theta,\mathcal{O}}(x) = c_{\theta_{x,\omega},\mathcal{O}}$ for every $\mathcal{O} \in Nil(\mathfrak{g}_x)$.

Now if θ is a quasi-character of $G(F)$ that is $Z_G(F)$-invariant, then

$$c_{\theta,\mathcal{O}}(zx) = c_{\theta,\mathcal{O}}(x)$$

for all $z \in Z_G$. For ω as above, we can define a quasi-character on $\mathfrak{g}_x(F)$ that is invariant by $\mathfrak{z}_{\mathfrak{g}}(F)$, which is still denoted by $\theta_{x,\omega}$, to be

$$\theta_{x,\omega}(X) = \begin{cases} \theta(x\exp(X')), & \text{if } X = X' + Z, X' \in \omega, Z \in \mathfrak{z}_{\mathfrak{g}}(F); \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

3.7 The Localization of θ_f

In this section, we discuss the localization of the quasi-character θ_f, which will be used in the localization of the trace formula in Chapter 9. Some results of this section will also be used in Chapter 11 when we change the truncated function in the trace formula. For $x \in G_{ss}(F)$, recall that $\mathfrak{g}_{x,0}$ is the subspace of elements in $\mathfrak{g}_x$ whose trace is zero. Suppose $\mathfrak{g}_{x,0} = \mathfrak{g}'_x \oplus \mathfrak{g}''$ where $\mathfrak{g}'_x$ and $\mathfrak{g}''$ are the Lie algebras of some connected reductive groups (See Section 9.3). For any element $X \in \mathfrak{g}_{x,0}(F)$, it can be decomposed as $X = X' + X''$ for $X' \in \mathfrak{g}'_x$ and $X'' \in \mathfrak{g}''$. We denote by $f \to f^\sharp$ the partial Fourier transform for $f \in C_x^\infty(\mathfrak{g}_{x,0}(F))$ with respect to X''. i.e.

$$f^\sharp(X) = \int_{\mathfrak{g}''(F)} f(X' + Y'')\psi(< Y'', X'' >)dY''. \tag{3.12}$$

Let ω be a good neighborhood of 0 in $\mathfrak{g}_x$. We can also view ω as an neighborhood of 0 in $\mathfrak{g}_{x,0}$ by considering its image in $\mathfrak{g}_{x,0}$ under the projection $\mathfrak{g}_x \to \mathfrak{g}_{x,0}$. If $f \in C_c^\infty(Z_G(F)\backslash G(F))$, for $g \in G(F)$, define ${}^g f_{x,\omega} \in C_c^\infty(\mathfrak{g}_{x,0}(F))$ by

$${}^g f_{x,\omega}(X) = \begin{cases} f(g^{-1}x\exp(X)g), & \text{if } X \in \omega; \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

Also define

$${}^g f_{x,\omega}^\sharp = ({}^g f_{x,\omega})^\sharp. \tag{3.14}$$

Note that for $X \in \mathfrak{g}_{x,0}(F)$, $X \in \omega$ means there exist $X' \in \omega$ and $Z \in \mathfrak{z}_\mathfrak{g}(F)$ such that $X = X' + Z$. It follows that the value $f(g^{-1}x\exp(X)g)$ is just $f(g^{-1}x\exp(X')g)$, which is independent of the choice of X' and Z.

If M is a Levi subgroup of G containing the given x, fix a hyperspecial open compact subgroup K with respect to M. If $P = MU \in \mathcal{P}(M)$, for $f \in C_c^\infty(Z_G(F)\backslash G(F))$, define the functions $\varphi[P, f], \varphi^\sharp[P, f]$ and $J^\sharp_{M,x,\omega}(\cdot, f)$ on $\mathfrak{m}_{x,0}(F) \cap \mathfrak{g}_{x,reg}(F)$ by

$$\varphi[P, f](X) = D^{G_x}(X)^{1/2} D^{M_x}(X)^{-1/2} \int_{U(F)} {}^u f_{x,\omega}(X)du, \tag{3.15}$$

$$\varphi^\sharp[P, f](X) = D^{G_x}(X)^{1/2} D^{M_x}(X)^{-1/2} \int_{U(F)} {}^u f^\sharp_{x,\omega}(X)du, \tag{3.16}$$

and

$$J^{\sharp}_{M,x,\omega}(X,f) = D^{G_x}(X)^{1/2} \int_{G_{x,X}(F)\backslash G(F)} {}^g f^{\sharp}_{x,\omega}(X) v_M(g) dg. \tag{3.17}$$

The following two lemmas are proved in Sections 5.4 and 5.5 of [W10], which will be used in the localization of the trace formula. The second lemma will also be used in Section 11 when we change the truncated function in the trace formula.

Lemma 3.7.1. *The following hold.*

1. *The three integrals above are absolutely convergent.*
2. *The function $\varphi[P,f]$ and $\varphi^{\sharp}[P,f]$ can be extended to elements in $C_c^{\infty}(\mathfrak{m}_{x,0}(F))$ and we have $(\varphi[P,f])^{\sharp} = \varphi^{\sharp}[P,f]$.*
3. *The function $X \to J^{\sharp}_{M,x,\omega}(X,f)$ is invariant under $M_x(F)$-conjugation, and has a compactly support modulo conjugation. Further, it is locally constant on $\mathfrak{m}_{x,0}(F) \cap \mathfrak{g}_{x,reg}(F)$, with the property that there exist $c > 0$ and an integer $k \geq 0$ such that*

$$\mid J^{\sharp}_{M,x,\omega}(X,f) \mid \leq c(1+ \mid \log(D^{G_x}(X)) \mid)^k$$

for every $X \in \mathfrak{m}_{x,0} \cap \mathfrak{g}_{x,reg}(F)$.

Lemma 3.7.2. *Suppose f is strongly cuspidal.*

1. *If $P \neq G$, the function $\varphi[P,f]$ and $\varphi^{\sharp}[P,f]$ are zero.*
2. *The function $J^{\sharp}_{M,x,\omega}(\cdot,f)$ does not depend on the choice of K. It is zero if $A_{M_x} \neq A_M$. For every $y \in G(F)$ and $X \in \mathfrak{m}_{x,0} \cap \mathfrak{g}_{x,reg}(F)$, we have*

$$J^{\sharp}_{M,x,\omega}(X,f) = J^{\sharp}_{M,x,\omega}(X, {}^y f).$$

For $f \in C_c^{\infty}(Z_G(F)\backslash G(F))$ strongly cuspidal, we define a function $\theta_{f,x,\omega}$ on $(\mathfrak{g}_{x,0})_{reg}$ by

$$\theta_{f,x,\omega}(X) = \begin{cases} \theta_f(x\exp(X)), & \text{if } X \in \omega; \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

If $X \in (\mathfrak{g}_{x,0})_{reg}$, let $M(X)$ be the centralizer of $A_{G_{x,X}}$ in G. We define

$$\theta^{\sharp}_{f,x,\omega}(X) = (-1)^{a_{M(X)}-a_G} \nu(G_{x,X})^{-1} D^{G_x}(X)^{-1/2} J^{\sharp}_{M(X),x,\omega}(X,f) \tag{3.19}$$

By the lemma above this is independent of the choice of K. From the discussion of θ_f, we have a similar lemma:

Lemma 3.7.3. *The functions $\theta_{f,x,\omega}$ and $\theta^\sharp_{f,x,\omega}$ are invariant under $G_x(F)$-conjugation, compactly supported modulo conjugation, locally integrable on $\mathfrak{g}_{x,0}(F)$, and locally constant on $\mathfrak{g}_{x,0,reg}(F)$.*

The next result about $\theta_{f,x,\omega}$ and $\theta^\sharp_{f,x,\omega}$ is proved in Section 5.8 of [W10]. It tells us that $\theta^\sharp_{f,x,\omega}$ is the partial Fourier transform of $\theta_{f,x,\omega}$ with respect to X''.

Proposition 3.7.4. *If $f \in C_c^\infty(Z_G(F)\backslash G(F))$ is strongly cuspidal, then $\theta^\sharp_{f,x,\omega}$ is the partial Fourier transform of $\theta_{f,x,\omega}$ in the sense that, for every $\varphi \in C_c^\infty(\mathfrak{g}_{x,0}(F))$, we have*

$$\int_{\mathfrak{g}_{x,0}(F)} \theta^\sharp_{f,x,\omega}(X)\varphi(X)dX = \int_{\mathfrak{g}_{x,0}(F)} \theta_{f,x,\omega}(X)\varphi^\sharp(X)dX. \tag{3.20}$$

3.8 Pseudo Coefficients

In this section we assume that $G = \mathrm{GL}_n(D)$ for some division algebra D/F. Let π be a discrete series of $G(F)$ with central character χ. For $f \in C_c^\infty(Z_G(F)\backslash G(F), \chi^{-1})$, we say f is a pseudo coefficient of π if the following conditions hold.

- $\mathrm{tr}(\pi(f)) = 1$.
- For all $\sigma \in \Pi_{temp}(G,\chi)$ with $\sigma \neq \pi$, we have $\mathrm{tr}(\sigma(f)) = 0$.

Lemma 3.8.1. *For all discrete series π of $G(F)$ with central character χ, the pseudo coefficients of π exist. Moreover, all pseudo coefficients are strongly cuspidal.*

Proof. The existence of the pseudo coefficient is proved in [BDK]. Let f be a pseudo coefficient, we want to show that f is strongly cuspidal. By the definition of f, we know that for all proper parabolic subgroup $P = MU$ of G, and for all tempered representations τ of $L(F)$, we have $\mathrm{tr}(\pi'(f)) = 0$ where $\pi' = I_P^G(\tau)$. Then by Proposition 3.5.2, we know that f is strongly cuspidal. This proves the lemma. $\square$

Chapter 4

The Ginzburg-Rallis Model and its Reduced Models

In this chapter, we study the analytic and geometric properties of the Ginzburg-Rallis model. Geometrically, we show that it is a wavefront spherical variety. This gives us the weak Cartan decomposition. Analytically, we show it has polynomial growth as a homogeneous space. Then by applying all such properties, we prove some estimates for several integrals which will be used in later chapters. We will also discuss the reduced models associated to the Ginzburg-Rallis model coming from parabolic induction. This is a technical section, readers may assume the results in the section at the beginning and come back for the proof later.

4.1 The Ginzburg-Rallis Models

Let (G,R) be the pair (G,R) or (G_D,R_D) as in Chapter 1, and let $G_0 = M$. Then (G_0,H) is just the trilinear model of $GL_2(F)$ or $GL_1(D)$. We define a homomorphism $\lambda : U(F) \to F$ to be

$$\lambda(u(X,Y,Z)) = \mathrm{tr}(X) + \mathrm{tr}(Y).$$

Therefore the character ξ we defined in Chapter 1 can be written as $\xi(u) = \psi(\lambda(u))$ for $u \in U(F)$. Similarly, we can define λ on the Lie algebra of U. We also extend λ to $R(F)$ by making it trivial on $H(F)$.

Lemma 4.1.1. *1. The map $G \to R\backslash G$ has the norm descent property. For the definition of the norm descent property, see Section 18 of [K05], or Section 1.2 of [B15].*

2. The orbit of λ under the M-conjugation is a Zariski open subset in $(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}])^*$.

Proof. (1) Since the map is obviously G-equivariant, by Proposition 18.2 of [K05], we only need to show that it admits a section over a nonempty Zariski-open subset. Let $\bar{P} = M\bar{U}$ be the opposite parabolic subgroup of $P = MU$ with respect to M, and let P' be the subgroup of $\bar{P}$ that consists of elements in $\bar{P}$ whose M-part is of the form $(1, h_1, h_2)$ where $h_1, h_2 \in GL_2(F)$ or $GL_1(D)$. By the Bruhat decompostion, the map $\phi : P' \to R\backslash G$ is injective and the image is a Zariski open subset of $R\backslash G$. Then the composition of ϕ^{-1} and the inclusion $P' \hookrightarrow G$ is a section on $Im(\phi)$. This proves (1).

(2) Assume $G = GL_6(F)$. We can easily identify $(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}])^*$ with $M_2(F) \times M_2(F)$ where $M_2(F)$ are the two by two matrix over F. Then it is easy to see that the orbit of λ under the $M(F)$-conjugation is $GL_2(F) \times GL_2(F)$, which is a Zariski open subset. This proves (2) for the split case. The proof for the quaternion case is similar. □

4.2 The Spherical Pair (G, R)

We say a parabolic subgroup $\bar{Q}$ of G is good if $R\bar{Q}$ is a Zariski open subset of G. This is equivalent to say that $R(F)\bar{Q}(F)$ is open in $G(F)$ under the analytic topology.

Proposition 4.2.1. *1. There exist minimal parabolic subgroups of G that are good and they are all conjugated to each other by some elements in $H(F)$. If $\bar{P}_{min} = M_{min}\bar{U}_{min}$ is a good minimal parabolic subgroup, we have $R \cap \bar{U}_{min} = \{1\}$ and the complement of $R(F)\bar{P}_{min}(F)$ in $G(F)$ has zero measure.*

2. A parabolic subgroup $\bar{Q}$ of G is good if and only if it contains a good minimal parabolic subgroup.

3. Let $\bar{P}_{min} = M_{min}\bar{U}_{min}$ be a good minimal parabolic subgroup and $A_{min} = A_{M_{min}}$ be the split center of M_{min}, and set

$$A_{min}^{+} = \{a \in A_{min}(F) \mid\mid \alpha(a) \mid\geq 1 \; for \; any \; \alpha \in \Psi(A_{min}, \bar{P}_{min}).$$

Then we have

(a) $\sigma_0(h)+\sigma_0(a) \ll \sigma_0(ha)$ *for all* $a \in A_{min}^+$, $h \in R(F)$.

(b) $\sigma(h) \ll \sigma(a^{-1}ha)$ *and* $\sigma_0(h) \ll \sigma_0(a^{-1}ha)$ *for all* $a \in A_{min}^+$, $h \in R(F)$.

4. $(1),(2)$ *and* (3) *also hold for the pair* (G_0, H).

Proof. (1) We first show the existence of a good minimal parabolic subgroup. In the quaternion case, we can just choose the lower triangle matrices, which form a good minimal parabolic subgroup by the Bruhat decomposition. (Note that in this case the minimal parabolic subgroup is not a Borel subgroup since G is not split). In the split case, we first show that it is enough to find a good minimal parabolic subgroup for the pair (G_0, H). Let B_0 be a good minimal parabolic subgroup for the pair (G_0, H), since we are in the split case, B_0 is a Borel subgroup of G_0. Let $B = \bar{U}B_0$. It is a Borel subgroup of G. By the Bruhat decomposition, $\bar{U}P$ is open in G. Together with the fact that B_0 is a good Borel subgroup of (G_0, H), we know BR is open in G, which makes B a good minimal parabolic subgroup.

For the pair (G_0, H), let $B_0 = (B^+, B^-, B')$ where B^+ is upper triangle Borel subgroup of GL_2, B^- is lower triangle Borel subgroup of GL_2 and $B' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} B^- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is easy to see that $B^+ \cap B^- \cap B' = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}$, hence $B_0 \cap H = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}$. Then by comparing the dimensions, we know B_0 is a good minimal parabolic subgroup.

Now we need to show that two good minimal parabolic subgroups are conjugated to each other by some elements in $R(F)$. Let $\bar{P}_{min}$ be the good minimal parabolic subgroup defined above, and let $\bar{P}'_{min}$ be another good minimal parabolic subgroup. We can always find $g \in G(F)$ such that $g\bar{P}_{min}g^{-1} = \bar{P}'_{min}$. Let $\mathcal{U} = R\bar{P}_{min}$ and $\mathcal{Z} = G - \mathcal{U}$. If $g \in \mathcal{Z}$, then

$$R\bar{P}'_{min} = Rg\bar{P}_{min}g^{-1} \subset \mathcal{Z}g^{-1},$$

which is impossible since $R\bar{P}'_{min}$ is Zariski open and $\mathcal{Z}$ is Zariski closed. Hence $g \in \mathcal{U} \cap G(F) = \mathcal{U}(F)$. If $g \in R(F)\bar{P}_{min}(F)$, then we are done. So it is enough to show that

$$\mathcal{U}(F) = R(F)\bar{P}_{min}(F).$$

We have the following two exact sequence:

$$0 \to H^0(F, \bar{P}_{min}) \to H^0(F, R\bar{P}_{min}) \to H^0(F, R/R \cap \bar{P}_{min}),$$

$$0 \to H^0(F, R\cap\bar{P}_{min}) \to H^0(F, R) \to H^0(F, R/R\cap\bar{P}_{min}) \to H^1(F, R\cap\bar{P}_{min}) \to H^1(F, R).$$

Therefore it is enough to show that the map

$$H^1(F, R \cap \bar{P}_{min}) \to H^1(F, R) \tag{4.1}$$

is injective.

If G is split, by our construction, $R \cap \bar{P}_{min} = \mathrm{GL}_1$. Since $H^1(F, \mathrm{GL}_n) = \{1\}$ for any $n \in \mathbb{N}$, the map (4.1) is injective. If G is not split, by our construction, $R \cap \bar{P}_{min} = H$, $R/R \cap \bar{P}_{min} = U$. Then the map (4.1) lies inside the exact sequence

$$0 \to H^0(F, H) \to H^0(F, R) \to H^0(F, U) \to H^1(F, H) \to H^1(F, R)$$

It is easy to see that the map $H^0(F, R) \to H^0(F, U)$ is surjective, therefore (4.1) is injective. This finishes the proof.

For the rest part of (1), since we have already proved that two good minimal parabolic subgroups can be conjugated to each other by some elements in $R(F)$, it is enough to prove the rest part for the specific good minimal parabolic $\bar{P}_{min}$ we defined above, which is obvious from the construction of $\bar{P}_{min}$. This proves (1). The proof for the pair (G_0, H) is similar.

(2) Let $\bar{Q}$ be a good parabolic subgroup, and let $P_{min} \subset \bar{Q}$ be a minimal parabolic subgroup. Set

$$\mathcal{G} = \{g \in G \mid g^{-1} P_{min} g \textit{ is good}\}.$$

This is a Zariski open subset of G since it is the inverse image of the Zariski open subset $\{\mathcal{V} \in Gr_n(\mathfrak{g}) \mid \mathcal{V} + \mathfrak{r} = \mathfrak{g}\}$ of the Grassmannian variety $Gr_n(\mathfrak{g})$ under the morphism $g \in G \to g^{-1}\mathfrak{p}_{min} g \in Gr_n(\mathfrak{g})$, here $n = dim(P_{min})$. By (1), there exists a good minimal parabolic subgroup, hence $\mathcal{G}$ is non-empty. Since $\bar{Q}$ is good, $\bar{Q}R$ is a Zariski open subset, hence $\bar{Q}R \cap \mathcal{G} \neq \emptyset$. So we can find $\bar{q}_0 \in \bar{Q}$ such that $\bar{q}_0^{-1} P_{min} \bar{q}_0$ is a good parabolic subgroup. Let

$$\mathcal{Q} = \{\bar{q} \in \bar{Q} \mid \bar{q}^{-1} P_{min} \bar{q} \textit{ is good}\}.$$

Then we know $\mathcal{Q}$ is a non-empty Zariski open subset. Since $\bar{Q}(F)$ is dense in $\bar{Q}$, $\mathcal{Q}(F)$ is non-empty. Let $\bar{q}$ be an element of $\mathcal{Q}(F)$. Then the minimal parabolic subgroup $\bar{q}^{-1}P_{min}\bar{q}$ is good and is defined over F. This proves (2). The proof for the pair (G_0, H) is similar.

(3) By the first part of the proposition, two good minimal parabolic subgroups are conjugated to each other by some elements in $R(F)$. This implies that (a) and (b) do not depend on the choice of minimal parabolic subgroups. Hence we may use the minimal parabolic subgroup $\bar{P}_{min}$ defined in (1). Next we show that (a) and (b) do not depend on the choice of M_{min}. Let M_{min}, M'_{min} be two choices of Levi subgroup. Then there exists $\bar{u} \in \bar{U}_{min}(F)$ such that $M'_{min} = \bar{u}M_{min}\bar{u}^{-1}$ and $A'^{+}_{min} = \bar{u}A^{+}_{min}\bar{u}^{-1}$. Since for $a \in A^{+}_{min}$, $a^{-1}\bar{u}a$ is a contraction, the sets $\{a^{-1}\bar{u}a\bar{u}^{-1} \mid a \in A^{+}_{min}\}$ and $\{a^{-1}\bar{u}^{-1}a\bar{u} \mid a \in A^{+}_{min}\}$ are bounded. This implies

$$\begin{aligned} \sigma_0(h\bar{u}a\bar{u}^{-1}) &\sim \sigma_0(ha), \\ \sigma(\bar{u}a\bar{u}^{-1}h\bar{u}a\bar{u}^{-1}) &\sim \sigma(a^{-1}ha), \\ \sigma_0(\bar{u}a\bar{u}^{-1}h\bar{u}a\bar{u}^{-1}) &\sim \sigma_0(a^{-1}ha) \end{aligned}$$

for all $a \in A^{+}_{min}$ and $h \in R(F)$. Therefore (a) and (b) do not depend on the choice of M_{min}. We may choose

$$M_{min} = \{\mathrm{diag}(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & a_4 \end{pmatrix}, \begin{pmatrix} a_5 & a_5 - a_6 \\ 0 & a_6 \end{pmatrix}) \mid a_i \in F^{\times}\}$$

in the split case, and choose

$$M_{min} = \{\mathrm{diag}(b_1, b_2, b_3) \mid b_j \in D^{\times}\}$$

in the non-split case.

For part (a), let $h = uh_0$ for $u \in U(F)$ and $h_0 \in H(F)$. Then we know $\sigma_0(h) \ll \sigma_0(h_0) + \sigma_0(u)$ and $\sigma_0(ha) = \sigma_0(uh_0a) \gg \sigma_0(u) + \sigma_0(h_0a)$. As a result, we may assume that $h = h_0 \in H(F)$. If we are in the non-split case, $Z_H\backslash H(F)$ is compact, and the argument is trivial. In the split case, since the norm is K-invariant, by the Iwasawa decomposition, we may assume that h_0 is upper triangle. Then by using the same argument as above, we can get rid of the unipotent part. Hence we may assume that

$h_0 = diag(h_1, h_2)$ with $h_1, h_2 \in F^\times$. By our choice of M_{min},

$$a = diag(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & a_4 \end{pmatrix}, \begin{pmatrix} a_5 & a_5 - a_6 \\ 0 & a_6 \end{pmatrix}) = diag(A_1, A_2, A_3) \tag{4.2}$$

with $\mid a_2 \mid \leq \mid a_1 \mid \leq \mid a_3 \mid \leq \mid a_4 \mid \leq \mid a_5 \mid \leq \mid a_6 \mid$. Since we only consider σ_0, we may assume that $\Pi a_i = 1$ and $h_1 h_2 = 1$. (In general, after modulo the center, we can not make determinant equal to 1, there should be some square class left. But we are talking about majorization, the square class will not effect our estimation.) In order to make the argument hold for the pair (G_0, H), here we only assume that $\mid a_2 \mid \leq \mid a_1 \mid, \mid a_3 \mid \leq \mid a_4 \mid, \mid a_5 \mid \leq \mid a_6 \mid$. It is enough to show that

$$\sigma(h_0) + \sigma(a) \ll \sigma(h_0 a). \tag{4.3}$$

In this case, $\sigma(h_0) \sim \log(max\{\mid h_1 \mid, \mid h_2 \mid\})$ and $\sigma(a) \sim \log(max\{\mid a_6 \mid, \mid a_4 \mid, \mid a_1 \mid\}) \sim \log(max\{\mid a_2^{-1} \mid\mid a_3^{-1} \mid\mid a_5^{-1} \mid\})$.

- If $h_2 \geq 1$, we have $\sigma(h_0) \sim \log(\mid h_2 \mid), \parallel h_0 A_3 \parallel \geq \mid a_6 h_2 \mid$, and $\parallel h_0 A_2 \parallel \geq \mid a_4 h_2 \mid$. So if $max\{\mid a_6 \mid, \mid a_4 \mid, \mid a_1 \mid\} = \mid a_6 \mid \; or \; \mid a_4 \mid$, (4.3) holds. By the same argument, if $max\{\mid a_2^{-1} \mid\mid a_3^{-1} \mid\mid a_5^{-1} \mid\} = \mid a_3^{-1} \mid \; or \; \mid a_5^{-1} \mid$, (4.3) also holds. Now the only case left is $max\{\mid a_6 \mid, \mid a_4 \mid, \mid a_1 \mid\} = \mid a_1 \mid$ and $max\{\mid a_2^{-1} \mid\mid a_3^{-1} \mid\mid a_5^{-1} \mid\} = \mid a_2^{-1} \mid$.
  - If $\mid a_6 \mid \geq 1$, then $\parallel h_0 A_3 \parallel \geq \mid a_6 h_2 \mid$ and $\parallel h_0 A_1 \parallel \geq \mid a_2^{-1} h_2^{-1} \mid$. Hence $\parallel h_0 A_1 \parallel \parallel h_0 A_3 \parallel^2 \geq \mid a_2^{-1} a_6^2 h_2 \mid \geq \mid a_2^{-1} h_2 \mid$. In particular, (4.3) holds.
  - If $\mid a_6 \mid < 1$, then $\mid a_5 \mid < 1$. In this case, $\parallel h_0 A_3 \parallel \geq \mid a_5^{-1} h_2 \mid$ and $\parallel h_0 A_1 \parallel \geq \mid a_1 h_2^{-1} \mid$. Hence $\parallel h_0 A_1 \parallel \parallel h_0 A_3 \parallel^2 \geq \mid a_5^{-2} a_1 h_2 \mid \geq \mid a_1 h_2 \mid$. In particular, (4.3) holds.
- If $h_1 \geq 1$, the argument is similar as above, we will skip it here.

This finishes the proof of (a) for both the pair (G, R) and the pair (G_0, H).

For part (b), the argument for σ_0 is an easy consequence of the argument for σ, so we only prove the first one. Still let $h = u h_0$. By the definition of A_{min}^+, $a^{-1} u a$ is an extension of u (i.e. $\sigma(a^{-1} u a) \geq \sigma(u)$), so we can still reduce to the case $h = h_0 \in H(F)$. For the non-split case, the argument is trivial since $a^{-1} h_0 a = h_0$. For the split case, still let $a = diag(A_1, A_2, A_3)$ as above. It is enough to show that for any $h \in GL_2(F)$,

$$\parallel h \parallel \leq max\{\parallel A_i^{-1} h A_i \parallel, i = 1, 2, 3\}. \tag{4.4}$$

Let $h = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. We may assume that $det(h) \geq 1$. Then $\parallel h \parallel = max\{x_{ij}\}$. If $\parallel h \parallel = x_{11}, x_{21}$ *or* x_{22}, it is easy to see that $\parallel h \parallel \leq \parallel A_1^{-1} h A_1 \parallel$. If $\parallel h \parallel = x_{12}$, then $\parallel h \parallel \leq \parallel A_2^{-1} h A_2 \parallel$. Therefore (4.4) holds, and this finishes the proof of (b).

(4) is already covered in the proof of (1), (2) and (3). $\square$

The above proposition tells us that $X = R\backslash G$ is a spherical variety of G and $X_0 = H\backslash G_0$ is a spherical variety of G_0. In [SV], the authors have introduced the notion of wavefront spherical variety. In the next proposition, we are going to show that X_0 is a wavefront spherical variety of G_0. We need to use this result for the weak Cartan decomposition of (G, R) and (G_0, H).

Proposition 4.2.2. *X_0 is a wavefront spherical variety of G_0.*

Proof. It's is enough to show that the little Weyl group W_{X_0} of X_0 is equal to the Weyl group of G_0, which is $(\mathbb{Z}/2\mathbb{Z})^3$. Here we use the method introduced by Knop in [Knop95] to calculate the little Weyl group. To be specific, use the same notation as loc. cit., let $B = B_1 \times B_2 \times B_3$ be a Borel subgroup of G_0. Without loss of generality, we may assume that B_i is the upper triangular Borel subgroup of GL_2. Let $\mathfrak{B}(X_0)$ be the set of all non-empty, closed, irreducible, B-stable subsets of X_0. It is easy to see that there is a bijection between $\mathfrak{B}(X_0)$ and the set of all non-empty, closed, irreducible, H_0-stable subsets of $G_0/B \simeq (\mathbb{P}^1)^3$. And we can easily write down such orbits: $(\mathbb{P}^1)^3$, X_{12}, X_{13}, X_{23} and Y where $X_{ij} = \{(a_1, a_2, a_3) \in (\mathbb{P}^1)^3 | a_i = a_j\}$ and $Y = \{(a_1, a_2, a_3) \in (\mathbb{P}^1)^3 | a_1 = a_2 = a_3\}$. Therefore $\mathfrak{B}(X_0)$ contain five elements

$$\mathfrak{B}(X_0) = \{X_0, Y_1, Y_2, Y_3, Z\} \tag{4.5}$$

where Z is the orbit of the identity element under the action of B, so it is an irreducible subset of codimension 2. And all Y_i's are closed, irreducible, B-stable subsets of codimension 1, with $Y_1 = \{H\backslash(g, g'b, g') \mid b \in B_2, g, g' \in \mathrm{GL}_2\}$, $Y_2 = \{H\backslash(gb, g', g) \mid b \in B_1, g, g' \in \mathrm{GL}_2\}$, and $Y_3 = \{H\backslash(g, gb, g') \mid b \in B_2, g, g' \in \mathrm{GL}_2\}$. Now we study the action of the Weyl group $W = W_{G_0}$ of G_0 on the set $\mathfrak{B}(X_0)$.

Let $\Delta(G_0) = \{\alpha_1, \alpha_2, \alpha_2\}$ be the set of simple roots of G_0 with respect to the Borel subgroup B, here α_i is the simple root of the i-th GL_2 with respect to B_i. For $i = 1, 2, 3$, let $w_i \in W$ be the simple reflection associated to α_i, and P_i be the corresponding minimal parabolic subgroup of G_0 containing B (i.e. P_i has B_j in the j-th component for $i \neq j$, and has GL_2 in the i-th component). Then we know W is generated by w_i's, hence it is enough to study the action of w_i on $\mathfrak{B}(X_0)$.

We first consider the action of w_1. It is easy to see that there are two non-empty, closed, irreducible, P_1-stable subsets of X_0: one is Y_1, the other one is X_0. Let

$$\mathfrak{B}(Y_1, P) = \{A \in \mathfrak{B}(X) \mid P_1 A = Y_1\}$$

and

$$\mathfrak{B}(X_0, P) = \{A \in \mathfrak{B}(X) \mid P_1 A = X_0\}.$$

We have $\mathfrak{B}(Y_1, P) = \{Y_1, Z\}$ and $\mathfrak{B}(X_0, P) = \{Y_2, Y_3, X_0\}$. By Theorem 4.2 of [Knop95], the action of w_1 on $\mathfrak{B}(X_0)$ is given by

$$w_1 \cdot X_0 = X_0, w_1 \cdot Y_1 = Y_1, w_1 \cdot Y_2 = Y_3, w_1 \cdot Y_3 = Y_2, w_1 \cdot Z = Z.$$

Similarly we can get the action of w_2 and w_3:

$$w_2 \cdot X_0 = X_0, w_2 \cdot Y_1 = Y_3, w_2 \cdot Y_2 = Y_2, w_2 \cdot Y_3 = Y_1, w_2 \cdot Z = Z;$$

$$w_3 \cdot X_0 = X_0, w_3 \cdot Y_1 = Y_2, w_3 \cdot Y_2 = Y_1, w_3 \cdot Y_3 = Y_3, w_3 \cdot Z = Z.$$

Hence the isotropy group of X_0 is W. By Theorem 6.2 of [Knop95], the little Weyl group W_{X_0} is just W, therefore X_0 is a wavefront spherical variety of G_0. □

We need the weak Cartan decomposition for X_0 and X. Let $\bar{P}_0 = M_0 \bar{U}_0$ be a good minimal parabolic subgroup of G_0, and let $A_0 = A_{M_0}$ be the maximal split central torus of M_0. Let

$$A_0^+ = \{a \in A_0(F) | \, |\alpha(a)| \geq 1, \ \forall \alpha \in \Psi(A_0, \bar{P}_0)\}.$$

Choose a good minimal parabolic subgroup $\bar{P}_{min} = \bar{P}_0 \bar{U} = M_{min} \bar{U}_{min}$ of G, and let P_{min} be its opposite with respect to M_{min}. Then we know $P_{min} \subset P$. Let Δ be the set of simple roots of $A_{min} = A_{M_{min}} = A_0$ in P_{min}, and let $\Delta_P = \Delta \cap \Psi(A_{min}, P)$ be the subset of simple roots appeared in $\mathfrak{u}$. For $\alpha \in \Delta_P$, let $\mathfrak{n}_\alpha$ be the corresponding root space.

Proposition 4.2.3. *1. There is a compact subset $\mathcal{K}_0 \subset G_0(F)$ such that*

$$G_0(F) = H(F) A_0^+ \mathcal{K}_0. \tag{4.6}$$

2. *There is a compact subset $\mathcal{K} \subset G(F)$ such that*

$$G(F) = R(F)A_0^+\mathcal{K}. \tag{4.7}$$

3. *The character ξ is nontrivial on $\mathfrak{n}_\alpha$ for all $\alpha \in \Delta_P$.*

Proof. We first prove that (1) implies (2). By the Iwasawa decomposition, there is a compact subgroup K of $G(F)$ such that $G(F) = P(F)K = U(F)M(F)K$. Now by part (1), there exists an open compact subset $\mathcal{K}_0$ of $G_0(F) = M(F)$ such that $G_0(F) = H(F)A_0^+\mathcal{K}_0$. Let $\mathcal{K} = \mathcal{K}_0 K$, then $R(F)A_0^+\mathcal{K} = U(F)H(F)A_0^+\mathcal{K}_0 K = U(F)M(F)K = G(F)$. This proves (2).

Now we prove (1): in the non-split case, $A_0^+ = Z_{G_0}$ and $Z_{G_0}\backslash G_0(F)$ is compact, hence (1) is trivial. In the split case, if $F = \mathbb{R}$, since (G_0, H) is a wavefront spherical variety, (2) follows from Theorem 5.13 of [KKSS]. If F is p-adic, we refer the readers to Appendix A for the explicit construction.

For part (3), it is easy to see that the statement is independent of the choice of the good minimal parabolic subgroup, so we still use the one defined in Proposition 4.2.1. Then (3) just follows from direct computation. □

To end this section, we will show that the homogeneous space $X = R\backslash G$ has polynomial growth. We first recall the definition for polynomial growth in [Ber88].

Definition 4.2.4. *We say a homogeneous space $X = R\backslash G$ of G has polynomial growth if it satisfies the following condition:*

For a fixed compact neighborhood K of the identity element in G, there exist constants $d, C > 0$ such that for every $t > 0$, the ball $B(t) = \{x \in X \mid r(x) \leq t\}$ can be covered by less than $C(1+t)^d$ many $K-$balls of the form $Kx, x \in X$. Here r is a function on X defined by $r(x) = \inf\{\sigma(g) \mid x = gx_0\}$ where $x_0 \in X$ is a fixed point.

Remark 4.2.5. *In our case, if we set $x_0 = 1$, then $r(x) = inf_{h\in R(F)}\sigma(hx)$. By Lemma 4.1.1, $r(x) = \sigma_{R\backslash G}(x)$.*

Lemma 4.2.6. *1. Let $\mathcal{K} \subset G(F)$ be a compact subset. We have $\sigma_{R\backslash G}(xk) \sim \sigma_{R\backslash G}(x)$ for all $x \in R(F)\backslash G(F), k \in \mathcal{K}$.*

2.

$$\sigma_{R\backslash G}(a) \sim \sigma_{Z_G\backslash G}(a) = \sigma_0(a) \tag{4.8}$$

for all $a \in A_0^+$, the last equation is just the definition of σ_0.

Proof. (1) is trivial. For (2), since $G \to R\backslash G$ has the norm descent property(Lemma 4.1.1), we may assume that

$$\sigma_{R\backslash G}(x) = \inf_{h\in R(F)} \sigma_G(hx). \tag{4.9}$$

Then we obviously have the inequality $\sigma_{R\backslash G}(g) \ll \sigma_0(g)$ for all $g \in G(F)$. So we only need to show that $\sigma_0(a) \ll \sigma_{R\backslash G}(a)$ for all $a \in A_0^+$. By applying (4.9), it is enough to show that for all $a \in A_0^+$ and $h \in R(F)$, we have

$$\sigma_0(a) \ll \sigma_0(ha). \tag{4.10}$$

We can write $h = uh_0$ for $u \in U(F), h_0 \in H(F)$. Since $\sigma_0(ug_0) \gg \sigma_0(g_0)$ for all $u \in U(F), g_0 \in G_0(F)$, we have $\sigma_0(ha) \gg \sigma_0(h_0a)$. So it is enough to show that for all $a \in A_0^+$ and $h_0 \in H(F)$, we have $\sigma_0(a) \ll \sigma_0(h_0a)$. This just follows from Proposition 4.2.1(3). This finishes the proof of (2). □

Proposition 4.2.7. *$R(F)\backslash G(F)$ has polynomial growth as $G(F)$-homogeneous space.*

Proof. By Proposition 4.2.3, there exists a compact subset $\mathcal{K} \subset G(F)$ such that $G(F) = R(F)A_0^+\mathcal{K}$. Since $R(F) \cap A_0^+ = Z_G(F)$, together with the lemma above, there exists a constant $c_0 > 0$ such that

$$B(t) \subset R(F)\{a \mid a \in A_0^+/Z_G(F), \sigma_0(a) \leq c_0 t\}\mathcal{K}$$

for all $t \geq 1$. Hence we only need to show that there exists a positive integer $N > 0$ such that for all $t \geq 1$, the subset $\{a \in A_0^+/Z_G(F) \mid \sigma_0(a) < t\}$ can be covered by less than $(1+t)^N$ subsets of the form C_0a with $a \in A_0^+$ and $C_0 \subset A_0^+$ is a compact subset with nonempty interior. This is trivial. □

4.3 Some Estimates

In the next two sections, we are going to prove several estimates for various integrals which will be used in later sections. The proof of some estimates are similar to the GGP case in [B15], we only include them here for completion.

Lemma 4.3.1. *1. There exists $\epsilon > 0$ such that the integral*

$$\int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(h_0) e^{\epsilon\sigma_0(h_0)} dh_0 \tag{4.11}$$

is absolutely convergent.

2. There exists $d > 0$ such that the integral

$$\int_{Z_R(F)\backslash R(F)} \Xi^G(h)\sigma_0(h)^{-d} dh \tag{4.12}$$

is absolutely convergent.

3. For all $\delta > 0$, there exists $\epsilon > 0$ such that the integral

$$\int_{Z_G(F)\backslash R(F)} \Xi^G(h) e^{\epsilon\sigma_0(h)} (1+ \mid \lambda(h) \mid)^{-\delta} dh \tag{4.13}$$

is absolutely convergent.

Proof. (1) If we are in the non-split case, $Z_H(F)\backslash H(F)$ is compact and the argument is trivial. If we are in the split case, $G_0 = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. By the definition of Ξ^{G_0}, for $h_0 \in H(F)$, $\Xi^{G_0}(h_0) = (\Xi^H(h_0))^3$. But since Ξ^H is the matrix coefficient of a tempered representation, it belongs to the space $L^{2+t}(Z_H(F)\backslash H(F))$ for any $t > 0$. Then we choose $\epsilon > 0$ small enough so that $e^{\epsilon\sigma_0(h_0)} \ll \Xi^H(h_0)^{-1/2}$. For such an ϵ, the integral (4.11) will be absolutely convergent.

(2) Let $d > 0$, by Proposition 2.8.3(iv), if d is sufficiently large,

$$\begin{aligned} \int_{Z_R(F)\backslash R(F)} \Xi^G(h)\sigma_0(h)^{-d} dh &= \int_{Z_R(F)\backslash H(F)} \int_{U(F)} \Xi^G(h_0 u)\sigma_0(h_0 u)^{-d} du dh_0 \\ &\ll \int_{Z_H(F)\backslash H(F)} \delta_P(h_0)^{1/2} \Xi^{G_0}(h_0) dh_0 \\ &= \int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(h_0) dh_0. \end{aligned}$$

And the last integral is absolutely convergent by (1).

(3) Since $\sigma_0(h_0 u) \ll \sigma_0(h_0)\sigma_0(u)$ for all $h_0 \in H(F)$ and $u \in U(F)$, by applying (1), it suffices to prove the following claim.

Claim 4.3.2. *For all $\delta > 0$ and $\epsilon_0 > 0$, there exists $\epsilon > 0$ such that the integral*

$$I^0_{\epsilon,\delta}(h_0) = \int_{U(F)} \Xi^G(uh_0)e^{\epsilon\sigma_0(u)}(1+ \mid \lambda(u) \mid)^{-\delta}du$$

is absolutely convergent for all $h_0 \in H(F)$, and we have

$$I^0_{\epsilon,\delta}(h_0) \ll \Xi^{G_0}(h_0)e^{\epsilon_0\sigma_0(h_0)}.$$

Given $\delta, \epsilon, \epsilon_0, b > 0$, we have $I^0_{\epsilon,\delta}(h_0) = I^0_{\epsilon,\delta,\leq b}(h_0) + I^0_{\epsilon,\delta,>b}(h_0)$ where

$$I^0_{\epsilon,\delta,\leq b}(h_0) = \int_{U(F)} 1_{\sigma_0\leq b}(u)\Xi^G(uh_0)e^{\epsilon\sigma_0(u)}(1+ \mid \lambda(u) \mid)^{-\delta}du$$

and

$$I^0_{\epsilon,\delta,>b}(h_0) = \int_{U(F)} 1_{\sigma_0>b}(u)\Xi^G(uh_0)e^{\epsilon\sigma_0(u)}(1+ \mid \lambda(u) \mid)^{-\delta}du.$$

For all $d > 0$, we have

$$I^0_{\epsilon,\delta,\leq b}(h_0) \leq e^{\epsilon b}b^d \int_{U(F)} \Xi^G(uh_0)\sigma_0(u)^{-d}du. \tag{4.14}$$

By Proposition 2.8.3(iv), we can choose $d > 0$ such that the last integral of (4.14) is essentially bounded by $\delta_P(h_0)^{-1/2}\Xi^M(h_0) = \Xi^{G_0}(h_0)$ for all $h_0 \in H(F)$. We fix such a $d > 0$, and then we have

$$I^0_{\epsilon,\delta,\leq b}(h_0) \ll e^{\epsilon b}b^d\Xi^{G_0}(h_0) \tag{4.15}$$

for all $h_0 \in H(F)$ and $b > 0$.

On the other hand, there exists $\alpha > 0$ such that $\Xi^G(gg') \ll e^{\alpha\sigma_0(g')}\Xi^G(g)$ for all $g, g' \in G(F)$. Therefore

$$I^0_{\epsilon,\delta,>b}(h_0) \ll e^{\alpha\sigma_0(h_0)-\sqrt{\epsilon}b} \int_{U(F)} \Xi^G(u)e^{(\epsilon+\sqrt{\epsilon})\sigma_0(u)}(1+ \mid \lambda(u) \mid)^{-\delta}du \tag{4.16}$$

for all $h_0 \in H(F)$ and $b > 0$. Assume that we can find $\epsilon > 0$ such that the last integral of (4.16) is convergent. Then by (4.15) and (4.16), we have

$$I^0_{\epsilon,\delta}(h_0) \ll e^{\epsilon b}b^d\Xi^{G_0}(h_0) + e^{\alpha\sigma_0(h_0)-\sqrt{\epsilon}b} \tag{4.17}$$

for all $h_0 \in H(F)$ and $b > 0$. Choose $\beta > 0$ such that $e^{-\beta\sigma_0(h_0)} \ll \Xi^{G_0}(h_0)$ for all $h_0 \in H(F)$, and then by letting $b = \frac{\alpha+\beta}{\sqrt{\epsilon}}\sigma_0(h_0)$ in (4.17), we have

$$\begin{aligned}
I^0_{\epsilon,\delta}(h_0) &\ll e^{\sqrt{\epsilon}(\alpha+\beta)\sigma_0(h_0)}(\frac{\alpha+\beta}{\sqrt{\epsilon}}\sigma_0(h_0))^d\Xi^{G_0}(h_0) + e^{\alpha\sigma_0(h_0)-(\alpha+\beta)\sigma_0(h_0)}(e^{\beta\sigma_0(h_0)}\Xi^{G_0}(h_0)) \\
&\ll e^{\sqrt{\epsilon}(\alpha+\beta+1)\sigma_0(h_0)}\Xi^{G_0}(h_0) + \Xi^{G_0}(h_0) \\
&\ll e^{\sqrt{\epsilon}(\alpha+\beta+1)\sigma_0(h_0)}\Xi^{G_0}(h_0)
\end{aligned}$$

for all $h_0 \in H(F)$. Note that α and β do not depend on the choice of ϵ. Hence we can always choose $\epsilon > 0$ small so that $\sqrt{\epsilon}(\alpha+\beta+1) < \epsilon_0$. This proves Claim 4.3.2.

So it remains to prove that we can find $\epsilon > 0$ such that the integral in (4.16) is absolutely convergent. If we are in the non-split case, P is a minimal parabolic subgroup of G, then this follows from Corollary B.3.1 of [B15]. If we are in the split case, it is easy to see that the convergence of the integral is independent of the choice of λ (under the M-conjugation), so we may temporarily let

$$\lambda(u(X,Y,Z)) = x_{12} + x_{21} + y_{12} + y_{21}$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Then we have a decomposition $\lambda = \lambda_+ - \lambda_-$ where

$$\lambda_+(u(X,Y,Z)) = x_{21} + y_{21}$$

and

$$\lambda_-(u(X,Y,Z)) = -x_{12} - y_{12}.$$

The additive character λ_+ is the restriction to U of a generic additive character of a maximal unipotent subgroup contained in P. In fact we can take the maximal unipotent subgroup to be the upper triangular unipotent matrix, and consider the additive character of the form $(x_{ij})_{1\leq i,j\leq 6} \to x_{12}+x_{23}+x_{34}+x_{45}+x_{56}$. By applying Corollary B.3.1 of [B15] again, we know the integral

$$\int_{U(F)} \Xi^G(u)e^{\epsilon\sigma_0(u)}(1+ \mid \lambda_+(u) \mid)^{-\delta} du \tag{4.18}$$

is convergent for ϵ small.

Fix an embedding $a : \mathbb{G}_m \hookrightarrow M$ given by $t \to diag(1,t,1,t,1,t)$. It is easy to see that $\lambda_+(a(t)ua(t)^{-1}) = t\lambda_+(u)$ and $\lambda_-(a(t)ua(t)^{-1}) = t^{-1}\lambda_-(u)$ for all $t \in \mathbb{G}_m$ and $u \in U(F)$. Let $\mathcal{U} \subset F^\times$ be a compact neighborhood of 1. For all $\epsilon > 0$, we have

$$\begin{aligned} & \int_{U(F)} \Xi^G(u)e^{\epsilon\sigma_0(u)}(1+ \mid \lambda(u) \mid)^{-\delta} du \\ \ll & \int_{U(F)} \Xi^G(u)e^{\epsilon\sigma_0(u)}(1+ \mid \lambda(a(t)ua(t)^{-1}) \mid)^{-\delta} du \\ = & \int_{U(F)} \Xi^G(u)e^{\epsilon\sigma_0(u)}(1+ \mid t\lambda_+(u) - t^{-1}\lambda_-(u) \mid)^{-\delta} du \end{aligned}$$

for all $t \in \mathcal{U}$. Integrating the above inequality over $\mathcal{U}$, we have

$$\begin{aligned} & \int_{U(F)} \Xi^G(u) e^{\epsilon\sigma_0(u)} (1+ \mid \lambda(u) \mid)^{-\delta} du \\ \ll & \int_{U(F)} \Xi^G(u) e^{\epsilon\sigma_0(u)} \int_{\mathcal{U}} (1+ \mid t\lambda_+(u) - t^{-1}\lambda_-(u) \mid)^{-\delta} dt du \end{aligned}$$

By Lemma B.1.1 of [B15], there exists $\delta' > 0$ only depends on $\delta > 0$ such that the last expression above is essentially bounded by

$$\int_{U(F)} \Xi^G(u) e^{\epsilon\sigma_0(u)} (1+ \mid t\lambda_+(u) \mid)^{-\delta'} du.$$

Then by (4.18), we can find $\epsilon > 0$ such that the integral on (4.16) is absolutely convergent. This finishes the proof of (3).

□

Lemma 4.3.3. *Let $\bar{P}_{min} = M_{min}\bar{U}_{min}$ be a good minimal parabolic subgroup of G.*

1. *For any $\delta > 0$, there exist $\epsilon > 0$ and $d > 0$ such that the integral*

$$I^1_{\epsilon,\delta}(m_{min}) = \int_{Z_R(F)\backslash R(F)} \Xi^G(hm_{min}) e^{\epsilon\sigma_0(h)} (1+ \mid \lambda(h) \mid)^{-\delta} dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$, and we have

$$I^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\bar{P}_{min}}(m_{min})^{-1/2} \sigma_0(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$.

2. *Assume that $Z_{G_0}(F)$ is contained in $A_{M_{min}}(F)$. Then for any $\delta > 0$, there exist $\epsilon > 0$ and $d > 0$ such that the integral*

$$\begin{aligned} I^2_{\epsilon,\delta}(m_{min}) \;=\; & \int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \\ & \Xi^G(hm_{min}) \Xi^G(h'hm_{min}) e^{\epsilon\sigma_0(h)} e^{\epsilon\sigma_0(h')} (1+ \mid \lambda(h') \mid)^{-\delta} dh' dh \end{aligned}$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$, and we have

$$I^2_{\epsilon,\delta}(m_{min}) \ll \delta_{\bar{P}_{min}}(m_{min})^{-1} \sigma_0(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$.

Proof. (1) Since $\Xi^G(g^{-1}) \sim \Xi^G(g), \sigma_0(g^{-1}) \sim \sigma_0(g)$ and $\lambda(h^{-1}) = -\lambda(h)$ for all $g \in G(F)$ and $h \in R(F)$, it is equivalent to prove the following Claim.

Claim 4.3.4. *For any $\delta > 0$, there exist $\epsilon > 0$ and $d > 0$, such that the integral*

$$J^1_{\epsilon,\delta}(m_{min}) = \int_{Z_R(F)\backslash R(F)} \Xi^G(m_{min}h)e^{\epsilon\sigma_0(h)}(1+ \mid \lambda(h) \mid)^{-\delta} dh$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$, and we have

$$J^1_{\epsilon,\delta}(m_{min}) \ll \delta_{\bar{P}_{min}}(m_{min})^{1/2}\sigma_0(m_{min})^d$$

for all $m_{min} \in M_{min}(F)$.

By Proposition 2.8.3(ii), there exists $d > 0$ such that

$$\begin{aligned} J^1_{\epsilon,\delta}(m_{min}) &\ll \delta_{\bar{P}_{min}}(m_{min})^{1/2}\sigma_0(m_{min})^d \\ &\quad \times \int_{Z_R(F)\backslash R(F)} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(h))^{1/2}\sigma_0(h)^d e^{\epsilon\sigma_0(h)}(1+ \mid \lambda(h) \mid)^{-\delta} dh \end{aligned}$$

for all $m_{min} \in M_{min}(F)$. Here $m_{\bar{P}_{min}} : G(F) \to \bar{P}_{min}(F)$ is the map induced by the Iwasawa decomposition. Since $\sigma_0(h)^d e^{\epsilon\sigma_0(h)} \ll e^{\epsilon'\sigma_0(h)}$ for all $\epsilon' > \epsilon > 0$, it is enough to prove that for ϵ small, the integral

$$\int_{Z_R(F)\backslash R(F)} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(h))^{1/2} e^{\epsilon\sigma_0(h)}(1+ \mid \lambda(h) \mid)^{-\delta} dh \tag{4.19}$$

is absolutely convergent. Since $\bar{P}_{min}$ is a good parabolic subgroup, we can find open compact neighborhoods of the identity $\mathcal{U}_K \subset K, \mathcal{U}_R \subset R(F)$ and $\mathcal{U}_{\bar{P}} \subset \bar{P}_{min}(F)$ such that $\mathcal{U}_K \subset \mathcal{U}_{\bar{P}}\mathcal{U}_R$. We have the estimates

$$e^{\epsilon\sigma_0(k_R h)} \ll e^{\epsilon\sigma_0(h)},\ (1+ \mid \lambda(k_R h) \mid)^{-\delta} \ll (1+ \mid \lambda(h) \mid)^{-\delta}$$

for all $h \in R(F)$ and $k_R \in \mathcal{U}_R$. Therefore

$$\begin{aligned} &\int_{Z_R(F)\backslash R(F)} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(h))^{1/2} e^{\epsilon\sigma_0(h)}(1+ \mid \lambda(h) \mid)^{-\delta} dh \\ \ll\ & \delta_{\bar{P}_{min}}(k_{\bar{P}})^{1/2} \int_{Z_R(F)\backslash R(F)} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(k_R h))^{1/2} e^{\epsilon\sigma_0(h)}(1+ \mid \lambda(h) \mid)^{-\delta} dh \\ =\ & \int_{Z_R(F)\backslash R(F)} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(k_{\bar{P}} k_R h))^{1/2} e^{\epsilon\sigma_0(h)}(1+ \mid \lambda(h) \mid)^{-\delta} dh \end{aligned}$$

for all $k_R \in \mathcal{U}_R$ and $k_{\bar{P}} \in \mathcal{U}_{\bar{P}}$. This implies

$$\begin{aligned}
&\int_{Z_R(F)\backslash R(F)} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(h))^{1/2} e^{\epsilon\sigma_0(h)}(1+\mid \lambda(h) \mid)^{-\delta} dh \\
\ll &\int_{Z_R(F)\backslash R(F)} \int_{\mathcal{U}_K} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(kh))^{1/2} dk e^{\epsilon\sigma_0(h)}(1+\mid \lambda(h) \mid)^{-\delta} dh \\
\ll &\int_{Z_R(F)\backslash R(F)} \int_{K} \delta_{\bar{P}_{min}}(m_{\bar{P}_{min}}(kh))^{1/2} dk e^{\epsilon\sigma_0(h)}(1+\mid \lambda(h) \mid)^{-\delta} dh.
\end{aligned}$$

By Proposition 2.8.3(iii), the inner integral above is equal to $\Xi^G(h)$. Then the convergence of (4.19) for ϵ small just follows from (3) of Lemma 4.3.1, this finishes the proof of (1).

(2) By changing the variable $h' \to h'h^{-1}$ in the integral, it is enough to show that for $\epsilon > 0$ small, the integral

$$\begin{aligned}
I^3_{\epsilon,\delta}(m_{min}) \;=\; &\int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \\
&\Xi^G(hm_{min})\Xi^G(h'm_{min}) e^{\epsilon\sigma_0(h)} e^{\epsilon\sigma_0(h')}(1+\mid \lambda(h') - \lambda(h) \mid)^{-\delta} dh' dh
\end{aligned}$$

is absolutely convergent for all $m_{min} \in M_{min}(F)$, and there exists $d > 0$ such that

$$I^3_{\epsilon,\delta}(m_{min}) \ll \delta_{\bar{P}_{min}}(m_{min})^{-1}\sigma_0(m_{min})^d \tag{4.20}$$

for all $m_{min} \in M_{min}(F)$. Let $a : \mathbb{G}_m(F) \to Z_{G_0}(F)$ be a homomorphism given by $a(t) = diag(t,t,1,1,t^{-1},t^{-1})$ in the split case, and $a(t) = diag(t,1,t^{-1})$ in the non-split case. It is easy to see that $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $h \in R(F)$ and $t \in \mathbb{G}_m(F)$. Let $\mathcal{U} \subset F^\times$ be an open compact neighborhood of 1. Since Z_{G_0} is in the center of M_{min}, by making the transform $h' \to a(t)^{-1}h'a(t)$, we have

$$\begin{aligned}
I^3_{\epsilon,\delta}(m_{min}) \;\ll\; &\int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min}) e^{\epsilon\sigma_0(h)} e^{\epsilon\sigma_0(h')} \\
&\times \int_{\mathcal{U}} (1+\mid \lambda(a(t)h'a(t)^{-1}) - \lambda(h) \mid)^{-\delta} dt dh' dh \\
= \;&\int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min}) e^{\epsilon\sigma_0(h)} e^{\epsilon\sigma_0(h')} \\
&\times \int_{\mathcal{U}} (1+\mid t\lambda(h') - \lambda(h) \mid)^{-\delta} dt dh' dh
\end{aligned}$$

for all $m_{min} \in M_{min}(F)$. By Lemma B.1.1 of [B15], there exists $\delta' > 0$, only depending on δ, such that the last integral above is essentially bounded by

$$\begin{aligned} & \int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \Xi^G(hm_{min})\Xi^G(h'm_{min}) \\ & e^{\epsilon\sigma_0(h)} e^{\epsilon\sigma_0(h')} (1+ \mid \lambda(h') \mid)^{-\delta'} (1+ \mid \lambda(h) \mid)^{-\delta'} dh' dh \\ = \; & I^1_{\epsilon,\delta'}(m_{min})^2 \end{aligned}$$

for all $m_{min} \in M_{min}(F)$. Therefore the inequality (4.20) follows from part (1), and this finishes the proof of (2). $\square$

4.4 The Harish-Chandra-Schwartz Spece of $R\backslash G$

Let $C \subset G(F)$ be a compact subset with nonempty interior. Define the function $\Xi_C^{R\backslash G}(x) = vol_{R\backslash G}(xC)^{-1/2}$ for $x \in R(F)\backslash G(F)$. If C' is another compact subset with nonempty interior, then $\Xi_C^{R\backslash G}(x) \sim \Xi_{C'}^{R\backslash G}(x)$ for all $x \in R(F)\backslash G(F)$. We will only use the function $\Xi_C^{R\backslash G}$ for majorization. From now on, we will fix a particular C, and set $\Xi^{R\backslash G} = \Xi_C^{R\backslash G}$. The next proposition gives the properties for the function $\Xi^{R\backslash G}$, which is quiet similar to Proposition 2.8.3 for the group case.

Proposition 4.4.1. *1. Let $\mathcal{K} \subset G(F)$ be a compact subset. We have $\Xi^{R\backslash G}(xk) \sim \Xi^{R\backslash G}(x)$ for all $x \in R(F)\backslash G(F)$ and $k \in \mathcal{K}$.*

2. Let $\bar{P}_0 = M_0\bar{U}_0$ be a good minimal parabolic subgroup of G_0, and let $A_0 = A_{M_0}$ be the split center of M_0. Set

$$A_0^+ = \{a_0 \in A_0(F) \mid\mid \alpha(a) \mid\geq 1 \; for \; all \; \alpha \in \Psi(A_0, \bar{P}_0)\}.$$

Then there exists $d > 0$ such that

$$\Xi^{G_0}(a)\delta_P(a)^{1/2}\sigma_{Z_{G_0}\backslash G_0}(a)^{-d} \ll \Xi^{R\backslash G}(a) \ll \Xi^{G_0}(a)\delta_P(a)^{1/2} \tag{4.21}$$

for all $a \in A_0^+$.

3. There exists $d > 0$ such that the integral

$$\int_{R(F)\backslash G(F)} \Xi^{R\backslash G}(x)^2 \sigma_{R\backslash G}(x)^{-d} dx$$

is absolutely convergent.

4. *For all* $d > 0$*, there exists* $d' > 0$ *such that*
$$\int_{R(F)\backslash G(F)} 1_{\sigma_{R\backslash G}\leq c}(x)\Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^d dx \ll c^{d'}$$
for all $c \geq 1$*.*

5. *There exist* $d > 0$ *and* $d' > 0$ *such that*
$$\int_{Z_R(F)\backslash R(F)} \Xi^G(x^{-1}hx)\sigma_0(x^{-1}hx)^{-d}dh \ll \Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^{d'}$$
for all $x \in R(F)\backslash G(F)$*.*

6. *For all* $d > 0$*, there exists* $d' > 0$ *such that*
$$\int_{Z_R(F)\backslash R(F)} \Xi^G(hx)\sigma_0(hx)^{-d'}dh \ll \Xi^{R\backslash G}(x)\sigma_{R\backslash G}(x)^{-d}$$
for all $x \in R(F)\backslash G(F)$*.*

7. *Let* $\delta > 0$ *and* $d > 0$*, then the integral*
$$\begin{aligned} I_{\delta,d}(c,x) \quad = \quad & \int_{Z_R(F)\backslash R(F)}\int_{Z_R(F)\backslash R(F)} 1_{\sigma_0\geq c}(h')\Xi^G(hx)\Xi^G(h'hx) \\ & \sigma_0(hx)^d\sigma_0(h'hx)^d(1+\mid\lambda(h')\mid)^{-\delta}dh'dh \end{aligned}$$
is absolutely convergent for all $x \in R(F)\backslash G(F)$ *and* $c \geq 1$*. Moreover, there exist* $\epsilon > 0$ *and* $d' > 0$ *such that*
$$I_{\delta,d}(c,x) \ll \Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^{d'}e^{-\epsilon c}$$
for all $x \in R(F)\backslash G(F)$ *and* $c \geq 1$*.*

Proof. The first one is trivial. For (2), let $\bar{P} = M\bar{U}$ be the parabolic subgroup opposite to P with respect to M. We fix some compact subsets with nonempty interior for the following groups
$$C_{\bar{U}} \subset \bar{U}(F), C_0 \subset G_0(F) = M(F), C_U \subset U(F).$$
By the Bruhat decomposition, $C = C_U C_0 C_{\bar{U}}$ is a compact subset of $G(F)$ with nonempty interior. By the definition of $\Xi^{R\backslash G}$, we have
$$\Xi^{R\backslash G}(g) \sim vol_{R\backslash G}(R(F)gC)^{-1/2}, \ \forall g \in G(F).$$

By the definition of Ξ^{G_0}, there exists $d > 0$ such that

$$\Xi^{G_0}(g_0)\sigma_{Z_{G_0}\backslash G_0}(g_0)^{-d} \ll vol_{G_0}(C_0 g_0 C_0)^{-1/2} \ll \Xi^{G_0}(g_0), \ \forall g_0 \in G_0(F).$$

So in order to prove (4.21), it is enough to show that

$$\delta_P(a)^{-1} vol_{G_0}(C_0 a C_0)^{-1/2} \sim vol_{R\backslash G}(R(F)aC)$$

for all $a \in A_0^+$. By the definition of C, we know

$$R(F)aC = R(F)aC_{\bar{P}}$$

where $C_{\bar{P}} = C_0 C_{\bar{U}}$. Thus we only need to prove

$$\delta_P(a)^{-1} vol_{G_0}(C_0 a C_0)^{-1/2} \sim vol_{R\backslash G}(R(F)aC_{\bar{P}}) \tag{4.22}$$

for all $a \in A_0^+$.

Let $C_H \subset H(F)$ be a compact subset with nonempty interior, and let $C_R = C_U C_H$. It is a compact subset of $R(F)$ with nonempty interior. We claim that

$$vol_{R\backslash G}(R(F)aC_{\bar{P}}) \sim vol_G(C_R a C_{\bar{P}}) \tag{4.23}$$

for all $a \in A_0^+$. In fact, we have

$$vol_G(C_R a C_{\bar{P}}) = \int_{R(F)\backslash G(F)} \int_{R(F)} 1_{C_R a C_{\bar{P}}}(hx) dh dx.$$

The inner integral above is nonzero if and only if $x \in R(F)aC_{\bar{P}}$. If this holds, the inner integral is equal to

$$\mathrm{vol}_R(R(F) \cap C_R a C_{\bar{P}} x^{-1}) = vol_R(C_R(R(F) \cap aC_{\bar{P}} x^{-1})).$$

Therefore in order to prove (4.23), it is enough to show that

$$vol_R(C_R(R(F) \cap aC_{\bar{P}} x^{-1})) \sim 1$$

for all $a \in A_0^+$ and $x \in aC_{\bar{P}}$. For such an x, $C_R \subset C_R(R(F) \cap aC_{\bar{P}} x^{-1})$, so we only need to show

$$vol_R(C_R(R(F) \cap aC_{\bar{P}} x^{-1})) \ll 1.$$

In order to prove this, it is enough to show that the set $R(F)\cap aC'_{\bar{P}}a^{-1}$ remains uniformly bounded for all $a \in A_0^+$, here $C'_{\bar{P}} = C_{\bar{P}}C_{\bar{P}}^{-1}$. Since $\bar{P}\cap R = H$, $R(F)\cap aC'_{\bar{P}}a^{-1} = H(F)\cap aC'_0a^{-1}$ where $C'_0 = C'_{\bar{P}}\cap G_0(F)$. For $h_0 \in H(F)\cap aC'_0a^{-1}$, $a^{-1}h_0a \in C'_0$ is bounded. By Proposition 4.2.1(3), $\sigma(h_0) \ll \sigma(a^{-1}h_0a)$. Hence $H(F)\cap aC'_0a^{-1}$ is uniformly bounded for $a \in A_0^+$, and this finishes the proof of (4.23).

Now by applying (4.23), (4.22) is equivalent to

$$\delta_P(a)^{-1}vol_{G_0}(C_0aC_0) \sim vol_G(C_RaC_{\bar{P}}),\ \forall a \in A_0^+. \tag{4.24}$$

By the definition of C_R and $C_{\bar{P}}$, $C_RaC_{\bar{P}} = C_U(C_HaC_0)C_{\bar{U}}$. Since we have a decomposition of the Haar measure on $G(F)$: $dg = \delta_P(g_0)^{-1}dudg_0d\bar{u}$ where du, dg_0 and $d\bar{u}$ are Haar measures on respectively $U(F), G_0(F)$ and $\bar{U}(F)$, we have

$$vol_G(C_U(C_HaC_0)C_{\bar{U}}) \sim \delta_P(a)^{-1}vol_{G_0}(C_HaC_0).$$

Hence the last thing to show is that for all $a_0 \in A_0^+$, we have

$$vol_{G_0}(C_0aC_0) \sim vol_{G_0}(C_HaC_0). \tag{4.25}$$

The inequality $vol_{G_0}(C_0aC_0) \gg vol_{G_0}(C_HaC_0)$ is trivial. For the other direction, since $H(F)\bar{P}_0(F)$ is open in $G_0(F)$, we may assume that $C_0 = C_HC_{\bar{P}_0}$ where $C_{\bar{P}_0}$ is a compact subset in $\bar{P}_0(F)$ with nonempty interior. By the definition of A_0^+, $a^{-1}C_{\bar{P}_0}a$ is uniformly bounded since the action on the unipotent part is a contraction and the action preserves the Levi part. Hence there exists a compact subset $C' \subset G_0(F)$ such that

$$a^{-1}C_{\bar{P}_0}aC_0 \subset C'$$

for all $a \in A_0^+$. This implies

$$vol_{G_0}(C_0aC_0) \ll vol_{G_0}(C_HaC') \ll vol_{G_0}(C_HaC_0)$$

for all $a \in A_0^+$. This finishes the proof of (4.25) and hence the proof of (2).

(3) Set $B(R) = \{x \in R(F)\backslash G(F) \mid \sigma_{R\backslash G}(x) < R\}$. By Proposition 4.2.7, there exists $N > 0$ such that for all $R \geq 1$, the subset $B(R)$ can be covered by less than

$(1+R)^N$ many subsets of the the form xC for $x \in R(F)\backslash G(F)$ and $C \subset G(F)$ be a compact subset with non-empty interior. Let

$$I(R,d) = \int_{B(R+1)\backslash B(R)} \Xi^{R\backslash G}(x)^2 \sigma_{R\backslash G}(x)^{-d} dx.$$

We have

$$\int_{R(F)\backslash G(F)} \Xi^{R\backslash G}(x)^2 \sigma_{R\backslash G}(x)^{-d} dx = \Sigma_{R=1}^{\infty} I(R,d). \tag{4.26}$$

Since for all $R \geq 1$, $B(R+1)\backslash B(R)$ can be covered by some subsets $x_1 C, \cdots, x_{k_R} C$ with $k_R \leq (R+2)^N$, we have

$$I(R,d) \leq \Sigma_{i=1}^{k_R} \int_{x_i C} \Xi^{R\backslash G}(x)^2 \sigma_{R\backslash G}(x)^{-d} dx \tag{4.27}$$

for all $d > 0$ and $R \geq 1$. Since C is compact, together with the definition of $\Xi^{R\backslash G}$, we have

$$\begin{aligned} & \int_{yC} \Xi^{R\backslash G}(x)^2 \sigma_{R\backslash G}(x)^{-d} dx \\ \ll \;& vol_{H\backslash G}(yC) \Xi^{R\backslash G}(y)^2 \sigma_{R\backslash G}(y)^{-d} \\ \ll \;& vol_{H\backslash G}(yC) vol_{H\backslash G}(yC)^{-1} \sigma_{R\backslash G}(y)^{-d} = \sigma_{R\backslash G}(y)^{-d} \end{aligned}$$

for all $y \in R(F)\backslash G(F)$. Combining with (4.27), we have

$$I(R,d) \ll \Sigma_{i=1}^{k_R} \sigma_{R\backslash G}(x_i)^{-d} \tag{4.28}$$

for all $d > 0$ and $R \geq 1$. Since $x_i C \cap (B(R+1)\backslash B(R)) \neq \emptyset$, $\sigma_{R\backslash G}(x_i) \gg R$. Combining with (4.28), we have

$$I(R,d) \ll R^{-d} k_R \leq (R+2)^N R^{-d}$$

for all $d > 0$ and $R \geq 1$. So once we let $d > N+1$, (4.26) is absolutely convergent. This finishes the proof of (3).

The proof of (4) is very similar to (3), we will skip it here. For (5), by the Cartan decomposition in Proposition 4.2.3, we may assume that $x \in A_0^+$. Then by applying part (2) and Lemma 4.2.6, we only need to show that there exists $d > 0$ such that for all $a \in A_0^+$, we have

$$\int_{Z_R(F)\backslash R(F)} \Xi^G(a^{-1}ha) \sigma_0(a^{-1}ha)^{-d} dh \ll \Xi^{G_0}(a)^2 \delta_P(a). \tag{4.29}$$

But we know

$$
\begin{aligned}
& \int_{Z_R(F)\backslash R(F)} \Xi^G(a^{-1}ha)\sigma_0(a^{-1}ha)^{-d}dh \\
= \; & \int_{Z_H(F)\backslash H(F)} \int_{U(F)} \Xi^G(a^{-1}h_0ua)\sigma_0(a^{-1}h_0ua)^{-d}dudh_0 \\
= \; & \delta_P(a) \int_{Z_H(F)\backslash H(F)} \int_{U(F)} \Xi^G(a^{-1}h_0au)\sigma_0(a^{-1}h_0au)^{-d}dudh_0.
\end{aligned}
$$

By Proposition 2.8.3(4), for $d > 0$ large, we have

$$
\int_{U(F)} \Xi^G(a^{-1}h_0au)\sigma_0(a^{-1}h_0au)^{-d}du \ll \Xi^{G_0}(a^{-1}h_0a)
$$

for all $a \in A_0(F)$ and $h_0 \in H(F)$. Thus for $d > 0$ large, the left hand side of (4.29) is essentially bounded by

$$
\delta_P(a) \int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(a^{-1}h_0a)dh_0.
$$

So in order to prove (4.29), it is enough to show that

$$
\int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(a^{-1}h_0a)dh_0 \ll \Xi^{G_0}(a)^2 \tag{4.30}
$$

for all $a \in A_0^+$. If we are in the non-split case, $A_0 = Z_{G_0}$, so $\Xi^{G_0}(a) = \Xi^{G_0}(1)$. Then (4.30) holds since $Z_H(F)\backslash H(F)$ is compact. In the split case, let $\mathcal{U}_{H(F)} \subset H(F)$ and $\mathcal{U}_{\bar{P}_0} \subset \bar{P}_0(F)$ be some compact neighborhoods of the identity. By the definition of A_0^+, the subsets $a^{-1}\mathcal{U}_{\bar{P}_0}a$ remain uniformly bounded as $a \in A_0^+$. So we have

$$
\int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(a^{-1}h_0a)dh_0 \ll \int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(a^{-1}p_1h_1h_0h_2p_2a)dh_0
$$

for all $a \in A_0^+$, $h_1, h_2 \in \mathcal{U}_H$ and $p_1, p_2 \in \mathcal{U}_{\bar{P}_0}$. Let K_0 be a maximal compact subgroup of $G_0(F)$. Since $\bar{P}_0$ is a good parabolic subgroup, there exists a compact neighborhood of the identity $\mathcal{U}_{K_0} \subset K_0$ such that $\mathcal{U}_{K_0} \subset \mathcal{U}_{\bar{P}_0}\mathcal{U}_H \cap \mathcal{U}_H\mathcal{U}_{\bar{P}_0}$. So we have

$$
\begin{aligned}
& \int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(a^{-1}h_0a)dh_0 \\
\ll \; & \int_{Z_H(F)\backslash H(F)} \int_{\mathcal{U}_{K_0}^2} \Xi^{G_0}(a^{-1}k_1h_0k_2a)dk_1dk_2dh_0 \\
\ll \; & \int_{Z_H(F)\backslash H(F)} \int_{K_0^2} \Xi^{G_0}(a^{-1}k_1h_0k_2a)dk_1dk_2dh_0
\end{aligned}
$$

for all $a \in A_0^+$. By Proposition 2.8.3(6), the last integral above is bounded by

$$\Xi^{G_0}(a)^2 \int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(h_0)dh_0$$

for all $a \in A_0^+$. Then (4.30) follows from Lemma 4.3.1(1) and this finishes the proof of (5).

For (6), by applying the same reduction as in (5), we only need to show that there exists $d' > 0$ such that for all $a \in A_0^+$, we have

$$\int_{Z_R(F)\backslash R(F)} \Xi^G(ha)\sigma_0(ha)^{-d'} dh \ll \delta_P(a)^{1/2}\Xi^{G_0}(a)\sigma_0(a)^{-d}.$$

Again we decompose $dh = dudh_0$ and by applying Proposition 2.8.3(4), we only need to show that for all $a \in A_0^+$, we have

$$\int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(h_0a)dh_0 \ll \Xi^{G_0}(a)\sigma_0(a)^{-d}.$$

Then using the same argument as (5), together with Proposition 2.8.3(6) and Proposition 4.2.1, we reduce to show that the integral

$$\int_{Z_H(F)\backslash H(F)} \Xi^{G_0}(h_0)dh_0$$

is absolutely convergent, which is just Lemma 4.3.1(1). This finishes the proof of (6).

For (7), by applying the same reduction as in (5), together with the fact that for all $d > 0$ and $\epsilon > 0$, we have $1_{\sigma_0 \geq c}(h)\sigma_0(h)^d \ll e^{\epsilon\sigma_0(h)}e^{-\epsilon c/2}$, we reduce to show that for all $\delta > 0$, there exist $d > 0$ and $\epsilon > 0$ such that for all $a \in A_0^+$, we have

$$\int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \Xi^G(ha)\Xi^G(h'ha)e^{\epsilon\sigma_0(h)}e^{\epsilon\sigma_0(h')}(1+ \mid \lambda(h') \mid)^{-\delta}dh'dh \tag{4.31}$$
$$\ll \delta_P(a)\Xi^{G_0}(a)^2\sigma_0(a)^d.$$

Let $\bar{P}_{min} = \bar{P}_0\bar{U}$ and $M_{min} = M_0$, then $\bar{P}_{min}$ is a good parabolic subgroup of G, and M_{min} is a Levi subgroup of it which contains A_0. By Lemma 4.3.3(2), there exists $\epsilon > 0$ and $d > 0$ such that

$$\int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \Xi^G(ha)\Xi^G(h'ha)e^{\epsilon\sigma_0(h)}e^{\epsilon\sigma_0(h')}(1+ \mid \lambda(h') \mid)^{-\delta}dh'dh$$

$$\ll \delta_{\bar{P}_{min}}(a)^{-1}\sigma_0(a)^d$$

for all $a \in A_0^+$. But we know $\delta_{\bar{P}_{min}}(a)^{-1} = \delta_P(a)\delta_{P_0}(a)$. By Proposition 2.8.3(1), $\delta_{P_0}(a) \ll \Xi^{G_0}(a)^2$ for all $a \in A_0^+$. Therefore the inequality (4.31) holds for such d and ϵ. This finishes the proof of (7). □

Lemma 4.4.2. *Let $\bar{Q} = M_Q\bar{U}_Q$ be a good parabolic subgroup of G, $R_{\bar{Q}} = R \cap \bar{Q}$, and let $G_{\bar{Q}} = \bar{Q}/U_{\bar{Q}}$ be the reductive quotient of $\bar{Q}$. Then we have*

1. *$R_{\bar{Q}} \cap U_{\bar{Q}} = \{1\}$, hence we can view $R_{\bar{Q}}$ as a subgroup of $G_{\bar{Q}}$. We also have $\delta_{\bar{Q}}(h_{\bar{Q}}) = \delta_{R_{\bar{Q}}}(h_{\bar{Q}})$ for all $h_{\bar{Q}} \in R_{\bar{Q}}(F)$.*

2. *There exists $d > 0$ such that the integral*
$$\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(h_{\bar{Q}})\sigma_0(h_{\bar{Q}})^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}}$$
is absolutely convergent. Moreover, if we are in the (G_0, H)-case (this means we replace the pair (G, R) in the statement by the pair (G_0, H)), for all $d > 0$, the integral
$$\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(h_{\bar{Q}})\sigma_0(h_{\bar{Q}})^{d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}}$$
is absolutely convergent.

3. *Let $\bar{P}_{min} = M_{min}\bar{U}_{min} \subset \bar{Q}$ be a good minimal parabolic subgroup of G, and let $A_{min} = A_{M_{min}}$, $A_{min}^+ = \{a \in A_{min}(F) |\ |\alpha(a)| \geq 1, \forall \alpha \in \Psi(A_{min}, \bar{P}_{min})\}$. Then there exists $d > 0$ such that*
$$\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(a^{-1}h_{\bar{Q}}a)\sigma_0(a^{-1}h_{\bar{Q}}a)^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}} \ll \Xi^{G_{\bar{Q}}}(a)^2$$
for all $a \in A_{min}^+$

Proof. (1) $R_{\bar{Q}} \cap U_{\bar{Q}} = \{1\}$ just follows from Proposition 4.2.1. For the second part, we only need to show that
$$\det(Ad(h_{\bar{Q}})\mid_{\bar{\mathfrak{q}}/\mathfrak{r}_{\bar{Q}}}) = 1$$
for all $h_{\bar{Q}} \in R_{\bar{Q}}(F)$. Since $\bar{Q}$ is a good parabolic subgroup, $\bar{\mathfrak{q}} + \mathfrak{r} = \mathfrak{g}$ and $\mathfrak{r}_{\bar{Q}} = \mathfrak{r} \cap \bar{\mathfrak{q}}$, so we have an isomorphism $\bar{\mathfrak{q}}/\mathfrak{r}_{\bar{Q}} \simeq \mathfrak{g}/\mathfrak{r}$. This implies
$$\det(Ad(h_{\bar{Q}})\mid_{\bar{\mathfrak{q}}/\mathfrak{r}_{\bar{Q}}}) = \det(Ad(h_{\bar{Q}})\mid_{\mathfrak{g}/\mathfrak{r}}) = \det(Ad(h_{\bar{Q}})\mid_{\mathfrak{g}})\det(Ad(h_{\bar{Q}})\mid_{\mathfrak{r}})^{-1}.$$

Since G and R are unimodular, $\det(Ad(h_{\bar{Q}}) \mid_{\mathfrak{g}}) = \det(Ad(h_{\bar{Q}}) \mid_{\mathfrak{r}}) = 1$ for all $h_{\bar{Q}} \in R_{\bar{Q}}(F)$. This finishes the proof of (1).

(2) By Proposition 4.2.1, we can find a good minimal parabolic subgroup $\bar{P}_{min} = M_{min}\bar{U}_{min} \subset \bar{Q}$. Let L be the Levi subgroup of Q containing M_{min}, we have $L \simeq G_{\bar{Q}}$. Let K be a maximal compact subgroup of $G(F)$ in good position with respect to L, and let $K_L = K \cap L(F)$. Define $\tau = I_{\bar{P}_{min}\cap L}^{L}(1)$ and $\pi = I_{\bar{Q}}^{G}(\tau) = I_{\bar{P}_{min}}^{G}(1)$. Let $(\, , \,)$ (resp. $(\, , \,)_\tau$) be the inner product on π (resp. τ). We fix $e_K \in \pi^\infty$ (resp. $e_{K_L} \in \tau^\infty$) to be the unique K-invariant(resp. K_L-invariant) vector. Then by the definition of the Harish-Chandra function, we may assume that

$$\Xi^G(g) = (\pi(g)e_K, e_K), \Xi^L(l) = (\tau(l)e_{K_L}, e_{K_L})_\tau, g \in G(F), l \in L(F). \tag{4.32}$$

So by choosing a suitable Haar measure, we have

$$\Xi^G(g) = \int_{\bar{Q}(F)\backslash G(F)} (e_K(g'g), e_K(g'))_\tau dg'.$$

Since $\bar{Q}$ is a good parabolic, by part(1) and Proposition 4.2.1(1), we have

$$\int_{\bar{Q}(F)\backslash G(F)} \varphi(g)dg = \int_{R_{\bar{Q}}(F)\backslash R(F)} \varphi(h)dh$$

for all $\varphi \in L^1(\bar{Q}(F)\backslash G(F), \delta_{\bar{Q}})$. So for all $g \in G(F)$, we have

$$\Xi^G(g) = \int_{R_{\bar{Q}}(F)\backslash R(F)} (e_K(hg), e_K(h))_\tau dh.$$

By Lemma 4.3.1(2), there exists $d > 0$ such that the integral

$$\begin{aligned} &\int_{Z_R(F)\backslash R(F)} \Xi^G(h)\sigma_0(h)^{-d}dh \\ = &\int_{Z_R(F)\backslash R(F)} \int_{R_{\bar{Q}}(F)\backslash R(F)} (e_K(h'h), e_K(h'))_\tau \sigma_0(h)^{-d}dh'dh \end{aligned}$$

converges. Since $(e_K(h'h), e_K(h'))_\tau$ equals some value of Ξ^L, it is positive, so the double integral above is absolutely convergent. By switching the order of the integral, changing the variable $h \mapsto h'^{-1}h$ and decomposing the integral over $Z_R(F)\backslash R(F)$ as a double integral over $R_{\bar{Q}}\backslash R(F)$ and $Z_R(F)\backslash R_{\bar{Q}}(F)$, we know the integral

$$\int_{(R_{\bar{Q}}(F)\backslash R(F))^2} \int_{Z_R(F)\backslash R_{\bar{Q}}(F)} (\tau(h_{\bar{Q}})e_K(h), e_K(h'))_\tau \sigma_0(h'^{-1}h_{\bar{Q}}h)^{-d} \delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2} dh_{\bar{Q}}dhdh'$$

is absolutely convergent. Here we also use the fact that $\delta_{\bar{Q}}(h_{\bar{Q}}) = \delta_{R_{\bar{Q}}}(h_{\bar{Q}})$. Then by the Fubini Theorem, there exist $h, h' \in R(F)$ such that the integral

$$\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} (\tau(h_{\bar{Q}})e_K(h), e_K(h'))_\tau \sigma_0(h'^{-1}h_{\bar{Q}}h)^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}}$$

is absolutely convergent. Let $h = luk, h' = l'u'k'$ be the Iwasawa decomposition with $l, l' \in L(F), u, u' \in U_{\bar{Q}}(F)$ and $k, k' \in K$. Then by (4.32), for all $h_{\bar{Q}} \in R_{\bar{Q}}(F)$, we have

$$(\tau(h_{\bar{Q}})e_K(h), e_K(h'))_\tau = \delta_{\bar{Q}}(l'l)^{1/2}\Xi^L(l'^{-1}h_{\bar{Q}}l).$$

For the given h, h', l, l' as above, $\Xi^L(h_{\bar{Q}}) \ll \Xi^L(l'^{-1}h_{\bar{Q}}l)$ and $\sigma_0(h'^{-1}h_{\bar{Q}}h) \ll \sigma_0(h_{\bar{Q}})$ for all $h_{\bar{Q}} \in R_{\bar{Q}}(F)$. So the integral

$$\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^L(h_{\bar{Q}})\sigma_0(h_{\bar{Q}})^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}}$$

is absolutely convergent. This finishes the first part of (2) since $\Xi^L = \Xi^{G_{\bar{Q}}}$. The second part of (2) just follows from the same argument except we use Lemma 4.3.1(1) instead of Lemma 4.3.1(2).

(3) By Proposition 4.2.1(3), for all $d > 0$ and $a \in A_{min}^+$, we have

$$\begin{aligned} &\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(a^{-1}h_{\bar{Q}}a)\sigma_0(a^{-1}h_{\bar{Q}}a)^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}} \\ \ll &\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(a^{-1}h_{\bar{Q}}a)\sigma_0(h_{\bar{Q}})^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}}. \end{aligned}$$

So we only need to prove that there exists $d > 0$ such that for all $a \in A_{min}^+$, we have

$$\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(a^{-1}h_{\bar{Q}}a)\sigma_0(h_{\bar{Q}})^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}} \ll \Xi^{G_{\bar{Q}}}(a)^2.$$

Let $\bar{P}_{min,\bar{Q}}$ be the image of $\bar{P}_{min}$ under the projection $Q \to G_{\bar{Q}}$, it is a minimal parabolic subgroup of $G_{\bar{Q}}$ and $\bar{P}_{min,\bar{Q}}R_{\bar{Q}}$ is open in $G_{\bar{Q}}$. By applying the same argument as in the proof of Proposition 4.4.1(5), we can show that

$$\begin{aligned} &\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(a^{-1}h_{\bar{Q}}a)\sigma_0(h_{\bar{Q}})^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}} \\ \ll &\ \Xi^{G_{\bar{Q}}}(a)^2 \int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(h_{\bar{Q}})\sigma_0(h_{\bar{Q}})^{-d}\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}} \end{aligned}$$

for all $a \in A_{min}^+$. Then we just need to choose $d > 0$ large so that part (2) holds. This finishes the proof of (3). $\square$

4.5 The Reduced Models

In this section, we will discuss the reduced models associated to the Ginzburg-Rallis model. With the notation as in the previous section, the reduced models are just the models $(G_{\bar{Q}}, R_{\bar{Q}})$ where $\bar{Q} = M_Q \bar{U}_Q$ runs over the good parabolic subgroups of G. This models will be used in later chapters. To be specific, we will assume by induction that the local relative trace formulas and the multiplicity formulas hold for all these reduced models. Then based on this assumption, we can prove the local relative trace formulas and the multiplicity formulas for the Ginzburg-Rallis model. The proof of both formulas for the reduced models are the same as the Ginzburg-Rallis model. In other words, we only need to apply the same arguments in this paper to the reduced models. We will skip the details.

Roughly speaking, we can define the multiplicities of the reduced models as follows. Let τ be an irreducible generic representation of $G_{\bar{Q}}(F)$ whose central character equals χ^2 on $Z_G(F)$, we define the multiplicity $m(\tau)$ to be the dimension of the Hom space

$$Hom_{R_{\bar{Q}}(F)}(\tau, (\omega \otimes \xi)|_{R_{\bar{Q}}(F)} \otimes \delta_{R_{\bar{Q}}}^{1/2}).$$

Note that as in Lemma 4.4.2, when we consider the reduced models, we need to add the extra modular character $\delta_{R_{\bar{Q}}}^{1/2}$.

For our application, we need to divide the reduced models into two categories. We say the model $(G_{\bar{Q}}, R_{\bar{Q}})$ is of Type I if it appears both in the split case (i.e. $G(F) = \mathrm{GL}_6(F)$) and the quaternion case (i.e. $G(F) = \mathrm{GL}_3(D)$). This is equivalent to say that the parabolic subgroup $\bar{Q}$ is of type $(4,2)$, $(2,4)$ or $(2,2,2)$ in the split case; and of type $(2,1)$, $(1,2)$ or $(1,1,1)$ in the quaternion case. All the rest reduced models are called Type II models. In particular, Type II models only appear in the split case.

For the rest of this section, we will write down all the Type I models, as well as all the Type II models associated to the maximal parabolic subgroups. For simplicity, we will use $(G, R = H \ltimes U)$ to denote these models instead of $(G_{\bar{Q}}, R_{\bar{Q}})$.

We first consider the Type I models. Note that the extra modular character $\delta_{R_{\bar{Q}}}^{1/2}$ will be trivial for these models.

- **If $\bar{Q}$ is of type $(2,2,2)$ (or of type $(1,1,1)$ in the quaternion case)**, we get the trilinear GL_2 models. To be specific, we take $\bar{Q} = \bar{P}$. It is easy to see that

$\bar{Q}$ is a good parabolic subgroup. Then the reduced model can be described as follows: $G(F) = (\mathrm{GL}_2(F))^3$ and $R(F) = H(F) = \mathrm{GL}_2(F)$ diagonally embedded into $G(F)$. Let π be an irreducible generic representation of $G(F)$ whose central character equals χ^2 on $Z_G(F)$, the multiplicity $m(\pi)$ is just the dimension of the Hom space

$$Hom_{H(F)}(\pi, \omega)$$

where $\omega(h) = \chi(\det(h))$ for all $h \in H(F)$. Similarly, we can define the quaternion version with $G_D(F) = (\mathrm{GL}_1(D))^3$ and $H_D(F) = \mathrm{GL}_1(D)$.

- **If $\bar{Q}$ is of type $(4,2)$ (or of type $(2,1)$ in the quaternion case)**, we get a model between the trilinear GL_2 model and the Ginzburg-Rallis model, we will call it the middle model in this paper. Up to a finite isogeny, this model is just the Gan-Gross-Prasad model for $\mathrm{SO}(6) \times \mathrm{SO}(3)$. To be specific, let $\bar{Q}$ be the parabolic subgroup of type $(4,2)$ and contains the lower Borel subgroup. Then we get the middle model defined as follows: $G = GL_4(F) \times GL_2(F)$ and $P = MU$ be the parabolic subgroup of $G(F)$ with the Levi part M isomorphic to $GL_2(F) \times GL_2(F) \times GL_2(F)$ (i.e. P is the product of the second $GL_2(F)$ and the parabolic subgroup $P_{2,2}$ of the first $GL_4(F)$). The unipotent radical U consists of elements of the form

$$u = u(X) := \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; X \in M_2(F). \tag{4.33}$$

The character ξ on U is defined to be $\xi(u(X)) = \psi(\mathrm{tr}(X))$. Let $H = \mathrm{GL}_2(F)$ diagonally embeded into M. As before, χ induces a character on $H(F)$ and this gives us a one-dimensional representation $\omega \otimes \xi$ of $R := H \ltimes U$. For a given irreducible generic representation π of G, assume that $\omega_\pi = \chi^2$ on $Z_H(F)$. Define the multiplicity $m(\pi)$ to be

$$m(\pi) = \dim \mathrm{Hom}_{R(F)}(\pi, \omega \otimes \xi).$$

This model can be thought as the "middle model" between the Ginzburg-Rallis model and the trilinear model of GL_2.

- **If $\bar{Q}$ is of type** $(2,4)$ **(or of type** $(1,2)$ **in the quaternion case)**, we will still get the middle model as in the previous case.

Then we consider the Type II models. We will only write down those models associated to the maximal parabolic subgroups, the rest models are similar to the maximal ones. **The most important feature of the Type II models is that every semisimple element in $R(F)$ is split. As a result, in the multiplicity formulas and the geometric side of the relative trace formulas for these models, we only have the germ at the identity element**. For details, see Chapter 5.

- **If $\bar{Q}$ is of type** $(3,3)$, choose $\bar{Q}$ to be the parabolic subgroup of type $(3,3)$ and contains the lower Borel subgroup. We get the following model: $G = \mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$ and $R = H \ltimes U \subset G$ is of the following form:

$$H = \{h(a,b,x) = \begin{pmatrix} a & 0 & 0 \\ x & b & 0 \\ 0 & 0 & a \end{pmatrix} \times \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & x & b \end{pmatrix} |a,b \in F^\times,\ x \in F\}$$

  and

$$U = \{u(x_1,x_2,y_1,y_2) = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} |x_1,x_2,y_1,y_2 \in F\}.$$

  The character $\omega \otimes \xi$ on $R(F)$ is given as follows:

$$\omega \times \xi : h(a,b,x)u(x_1,x_2,y_1,y_2) \mapsto \frac{|b|^{1/2}}{|a|^{1/2}} \chi(ab)\psi(x_1+y_2).$$

  For a given irreducible generic representation π of $G(F)$ whose central character equals χ^2 on $Z_H(F)$, define the multiplicity $m(\pi)$ to be

$$m(\pi) = \dim \mathrm{Hom}_{R(F)}(\pi, \omega \times \xi).$$

- **If $\bar{Q}$ is of type** $(5,1)$, choose $\bar{Q}$ to be the parabolic subgroup of type $(5,1)$ and contains the lower Borel subgroup. We get the following model: $G = \mathrm{GL}_5(F) \times \mathrm{GL}_1(F)$ and $R = H \ltimes U \subset G$ is of the following form:

$$H = \{h(a,b,x) = diag(\begin{pmatrix} a & 0 \\ x & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ x & b \end{pmatrix}, \begin{pmatrix} a \end{pmatrix}) \times \begin{pmatrix} b \end{pmatrix} |a,b \in F^\times,\ x \in F\}$$

and

$$U = \{u(X, Y_1, Y_2) = \begin{pmatrix} I_2 & X & Y_1 \\ 0 & I_2 & Y_2 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 \end{pmatrix} | X \in M_{2\times 2}(F),\ Y_1, Y_2 \in M_{1\times 2}(F)\}.$$

Let $Y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \end{pmatrix}$ for $i = 1, 2$. The character $\omega \otimes \xi$ on $R(F)$ is given as follows:

$$\omega \times \xi : h(a, b, x)u(X, Y_1, Y_2) \to \frac{|b|^{1/2}}{|a|^{1/2}}\chi(ab)\psi(\mathrm{tr}(X) + y_{21}).$$

For a given irreducible generic representation π of $G(F)$ whose central character equals χ^2 on $Z_H(F)$, define the multiplicity $m(\pi)$ to be

$$m(\pi) = \dim \mathrm{Hom}_{R(F)}(\pi, \omega \times \xi).$$

- **If $\bar{Q}$ is of type** $(1, 5)$, we will get the same model as in the $(5, 1)$ case.

Chapter 5

The Statement of the Trace Formula

In this chapter, we write down both sides of the trace formula. We also write down the Lie algebra version of the geometric side of the trace formula. In Section 5.1, we define all the ingredients of the geometric expansion. In Section 5.2, we will define a truncated function κ_N and state the trace formula. It is worth to mention that the truncated function will only be used in the geometric side. Then we will show that in order to prove the geometric side of the trace formula, it is enough to consider functions with trivial central character. In Section 5.3, we will state the Lie algebra version of the trace formula. Finally, in Section 5.4, we will talk about the trace formulas for the reduced models. By induction, we will assume all these trace formulas hold. **In this chapter, we will assume that F is a p-adic field.**

5.1 The Ingredients of the Geometric Side

From this section and on, unless otherwise specified, we consider the Ginzburg-Rallis model. This is to consider a pair (G, H), which is either $(GL_6(F), GL_2(F))$ or $(GL_3(D), GL_1(D))$. Let $P = MU$ be the parabolic subgroup of the form $\begin{pmatrix} A & X & Z \\ 0 & B & Y \\ 0 & 0 & C \end{pmatrix}$ where A, B, C belong to $GL_2(F)$ (the split case) or $GL_1(D)$ (the non-split case), and X, Y, Z belong to $M_2(F)$

(the split case) or D (the non-split case). We can diagonally embed H into M, and define the character ξ on $U(F)$ by

$$\xi\left(\begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix}\right) = \psi(a\mathrm{tr}(X) + b\mathrm{tr}(Y)) \tag{5.1}$$

for some $a, b \in F^{\times}$.

Definition 5.1.1. *We define a function Δ on $H_{ss}(F)$ by*

$$\Delta(x) = \mid \det((1 - ad(x)^{-1})_{|U(F)/U_x(F)}) \mid_F .$$

Similarly, we can define Δ on $\mathfrak{h}_{ss}(F)$ by

$$\Delta(X) = \mid \det((ad(X))_{|\mathfrak{u}(F)/\mathfrak{u}_x(F)}) \mid_F .$$

Let $\mathcal{T}$ be a subset of subtori of H defined as follows:

- If $H = \mathrm{GL}_2(F)$, then $\mathcal{T}$ contain the trivial torus $\{1\}$ and the non-split torus T_v for $v \in F^{\times}/(F^{\times})^2, v \neq 1$ where $T_v = \{\begin{pmatrix} a & bv \\ b & a \end{pmatrix} \in H(F) \mid a, b \in F,\ (a,b) \neq (0,0)\}$.
- If $H = \mathrm{GL}_1(D)$, then $\mathcal{T}$ contain the subtorus T_v for $v \in F^{\times}/(F^{\times})^2$ with $v \neq 1$, where $T_v \subset D$ is isomorphic to the quadratic extension $F(\sqrt{v})$ of F.

Let θ be a quasi-character on $Z_G(F)\backslash G(F)$, and let $T \in \mathcal{T}$. If $T = \{1\}$, we are in the split case. In this case, we have a unique regular nilpotent orbit $\mathcal{O}_{reg}$ in $\mathfrak{g}(F)$ and take $c_\theta(1) = c_{\theta,\mathcal{O}_{reg}}(1)$. If $T = T_v$ for some $v \in F^{\times}/(F^{\times})^2$ with $v \neq 1$, we take $t \in T_v$ to be a regular element (in $H(F)$). It is easy to see in both cases that $G_t(F)$ is F-isomorphic to $\mathrm{GL}_3(F_v)$ where $F_v = F(\sqrt{v})$ is the quadratic extension of F. Let $\mathcal{O}_v$ be the unique regular nilpotent orbit in $\mathfrak{gl}_3(F_v)$, and take $c_\theta(t) = c_{\theta,\mathcal{O}_v}(t)$.

Proposition 5.1.2. *The function c_θ is locally constant on $T_{reg}(F)$ (here regular means as an element in $H(F)$). And the function $t \to c_\theta(t)D^H(t)\Delta(t)$ is locally integrable on $T(F)$.*

The first part of the proposition follows from the definition. The rest of this section is to prove the second part. The idea of the proof comes from [W10]. If $T = \{1\}$, there is nothing to prove since the integral is just evaluation. If $T = T_v$ for some $v \in F^\times/(F^\times)^2$ with $v \neq 1$, since $c_\theta(t)D^H(t)\Delta(t)$ is locally constant on $T_{reg}(F)$, and is invariant under $Z_H(F)$, we only need to show that the function is locally integrable around $t = 1$.

We need some preparations. For a finite dimensional vector space V over F, and any integer $i \in \mathbb{Z}$, let $C_i(V)$ be the space of functions $\varphi : V \to \mathbb{C}$ such that

$$\varphi(\lambda v) = |\lambda|^i \varphi(v)$$

for every $v \in V$ and $\lambda \in (F^\times)^2$. Then we let $C_{\geq i}(V)$ be the space of functions that are linear combinations of functions in $C_j(V)$ for $j \geq i$. For $T = T_v$ and $i \in \mathbb{Z}$, define the space $C_{\geq i}(T)$ to be the functions f on $T_{reg}(F)$ such that there is a neighborhood ω of 0 in $\mathfrak{t}(F)$ and a function $\varphi \in C_{\geq i}(\mathfrak{t}_0(F))$ such that

$$f(\exp(X)) = \varphi(\bar{X})$$

for all $0 \neq X \in \omega$, here $\bar{X}$ is the projection of X in $\mathfrak{t}_0(F)$. Then by [W10, Lemma 7.4], if $f \in C_{\geq 0}(T)$, f is locally integrable around $t = 1$. Hence we only need to show that the function $t \to c_\theta(t)D^H(t)\Delta(t)$ lies inside the space $C_{\geq 0}(T)$.

Once we choose ω small enough, we have $D^H(\exp(X)) = D^H(X)$ and $\Delta(\exp(X)) = \Delta(X)$ for all $0 \neq X \in \omega$. Hence the function $t \to D^H(t)\Delta(t)$ lies inside the space $C_{\geq 8}(T)$ where $8 = \delta(H) + dim(U_X)$. Therefore in order to prove Proposition 5.1.2, it is enough to prove the following lemma.

Lemma 5.1.3. *With the notations above, the function $t \to c_\theta(t)$ belongs to the space $C_{\geq -8}(T)$.*

Proof. By Section 3.6, if we choose ω small enough, we have

$$c_\theta(\exp(X)) = c_{\theta_{1,\omega},\mathcal{O}_X}(X)$$

for all $0 \neq X \in \omega$. Here $\theta_{1,\omega}$ is the localization of θ at 1 defined in Section 3.6, and $\mathcal{O}_X$ is the unique regular nilpotent orbit in $\mathfrak{g}_X$. Since in a small neighborhood of $0 \in \mathfrak{g}_0(F)$, $\theta_{1,\omega}$ is a linear combination of $\hat{j}(\mathcal{O}, \cdot)$ where $\mathcal{O}$ runs over the nilpotent orbit in $\mathfrak{g}_0$. Hence we may assume that $\theta_{1,\omega} = \hat{j}(\mathcal{O}, \cdot)$ for some $\mathcal{O}$.

If $\mathcal{O}$ is regular, then we are in the split case (i.e. $G = \mathrm{GL}_6(F)$) and $\mathcal{O}$ is the unique regular nilpotent orbit in $\mathfrak{g}_0$. As a result, the distribution $\hat{j}(\mathcal{O}, \cdot)$ is induced from the Borel subgroup and hence only supported in the Borel subalgebra. But by our construction of $T = T_v$, for any $t \in T_{reg}(F)$, we can always find a small neighborhood of t in G such that any element in such a neighborhood does not belong to the Borel subalgebra. Therefore the function $c_\theta(t)$ is identically zero, and the function $t \to c_\theta(t)D^H(t)\Delta(t)$ is obviously locally integrable.

If $\mathcal{O}$ is not regular, by (2.4) and (3.4), the function $c_{\theta_{1,\omega},\mathcal{O}_X}(X)$ belongs to the space $C_{\frac{dim(\mathcal{O}_X)-dim(\mathcal{O})}{2}}(\mathfrak{t}_0)$. The dimension of $\mathcal{O}_X$ is equal to $\delta(G_X) = 12$. On the other hand, since $\mathcal{O}$ is not regular, $dim(\mathcal{O}) \leq \delta(G) - 2 = 28$. Hence the function $c_{\theta_{1,\omega},\mathcal{O}_X}(X)$ belongs to the space $C_{\geq -8}(\mathfrak{t}_0)$. This finishes the proof of the lemma, and hence the proof of Proposition 5.1.2. □

5.2 The Trace Formula

Let $f \in C_c^\infty(Z_G(F)\backslash G(F), \chi^{-2})$ be a strongly cuspidal function. For $g \in G(F)$, we define the function ${}^g f^\xi$ on $H(F)/Z_H(F)$ by

$$ {}^g f^\xi(x) = \int_{U(F)} f(g^{-1}xug)\xi(u)du. $$

This is a function belonging to $C_c^\infty(Z_H(F)\backslash H(F), \chi^{-2})$. Define

$$ I(f, g) = \int_{Z_H(F)\backslash H(F)} {}^g f^\xi(x)\omega(x)dx, \tag{5.2} $$

and for each $N \in \mathbb{N}$, define

$$ I_N(f) = \int_{U(F)H(F)\backslash G(F)} I(f, g)\kappa_N(g)dg. \tag{5.3} $$

Here κ_N is a characteristic function on $G(F)$ defined below, which is left $U(F)H(F)$-invariant, right K-invariant, and compactly supported modulo $U(F)H(F)$: If G is split (i.e. $G = GL_6(F)$), for $g \in G(F)$, let $g = umk$ be its Iwasawa-decomposition with $u \in U(F)$, $m \in M(F)$ and $k \in K$. Then m is of the form $\mathrm{diag}(m_1, m_2, m_3)$ with $m_i \in GL_2(F)$. For any $1 \leq i, j \leq 3$, let $m_i^{-1}m_j = \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix} k_{ij}$ be its Iwasawa decomposition. We define κ_N to be

$$\kappa_N(g) = \begin{cases} 1, & \text{if } \sigma(a_{ij}), \sigma(b_{ij}) \leq N, \sigma(c_{ij}) \leq (1+\epsilon)N; \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

Here $\epsilon > 0$ is a fixed positive real number. Note that we do allow some more freedom on the unipotent part, which will be used when we are trying to change our truncated function to the one given by Arthur in his local trace formula. For details, see Chapter 11. If G is not split (i.e. $G = GL_3(D)$), we still have the Iwasawa decomposition $g = umk$ with $m = \mathrm{diag}(m_1, m_2, m_3)$, and $m_i \in GL_1(D)$. We define κ_N to be

$$\kappa_N(g) = \begin{cases} 1, & \text{if } \sigma(m_i^{-1} m_j) \leq N; \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

It follows that the integral in (5.3) is absolutely convergent because the integrand is compactly supported. The distribution in our trace formula is just

$$\lim_{N \to \infty} I_N(f).$$

Remark 5.2.1. *In fact, later in Appendix B, we will show that the integral*

$$I(f) = \int_{U(F)H(F)\backslash G(F)} I(f,g)dg$$

is absolutely convergent. In other word, we have

$$\lim_{N \to \infty} I_N(f) = I(f).$$

However, if we include the integral defining $I(f,g)$ (i.e. (5.2)), the double integral will not be absolutely convergent, and this is the reason for us to introduce the truncated function κ_N. We will use the expression $\lim_{N\to\infty} I_N(f)$ to prove the geometric side, and we will use the expression $I(f)$ to prove the spectral side.

For each $T \in \mathcal{T}$, let c_f be the function c_{θ_f} defined in the last section. Define the geometric side of the trace formula to be

$$I_{geom}(f) = \sum_{T \in \mathcal{T}} |W(H,T)|^{-1} \nu(T) \int_{Z_G(F)\backslash T(F)} c_f(t) D^H(t) \Delta(t) \omega(h) dt. \tag{5.6}$$

Since for any $T \in \mathcal{T}$, $Z_G(F)\backslash T$ is compact, the absolute convergence of the integral above follows from Proposition 5.1.2.

For the spectral side, define

$$I_{spec}(f) = \int_{\Pi_{temp}(G,\chi^2)} \theta_f(\pi) m(\bar{\pi}) d\pi$$

where $\theta_f(\pi)$ is defined in Section 3.5 and $m(\bar{\pi})$ is the multiplicity for the Ginzburg-Rallis model. The trace formula is stated in the following theorem.

Theorem 5.2.2. *For every function $f \in C_c^\infty(Z_G(F)\backslash G(F), \chi^{-2})$ that is strongly cuspidal, the following holds:*

$$I_{spec}(f) = \lim_{N\to\infty} I_N(f) = I_{geom}(f). \tag{5.7}$$

The spectral expansion will be proved from Chapter 6 to Chapter 8, while the geometric expansion will be proved from Chapter 9 to Chapter 12.

For the rest of this section, we are going to reduce the proof of the geometric expansion to the case when the test function f has trivial central character.

Proposition 5.2.3. *If the geometric expansion*

$$\lim_{N\to\infty} I_N(f) = I_{geom}(f)$$

holds for every stronly cuspidal functions f with trivial central character, then it holds in general.

Proof. Let f be an arbitrary test functions in the trace formula (i.e. the central character does not need to be trivial). Note that both $I_{geom}(f)$ and $I_N(f)$ are linear on f. Since

$$Z_G(F)\backslash G(F)/\{g \in G(F) \mid \det(g) = 1\}$$

is finite, we can localize f such that f is supported on

$$Z_G(F)g_0\{g \in G(F) \mid \det(g) = 1\}$$

for some $g_0 \in G(F)$. Let $G_1(F) = \{g \in G(F) \mid \det(g) = 1\}$, which is $SL_6(F)$ or $SL_3(D)$. Fix a fundamental domain $X \subset G_1(F)$ of $G_1(F)/(Z_G(F) \cap G_1(F)) = G_1(F)/Z_{G_1}(F)$. We may choose X so that it is open in $G_1(F)$. It is easy to see that $Z_{G_1}(F)$ is finite.

By further localizing f we may assume that f is supported on $Z_G(F)g_0X$. Define a function $f' \in C_c^\infty(Z_G(F)\backslash G(F))$ to be

$$f'(g) = \begin{cases} f(g'), & \text{if } g = g'z,\ g' \in g_0X,\ z \in Z_G(F); \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

It is easy to see that f' is well defined and is strongly cuspidal, and can be viewed as the extension by trivial central character of the function $f\mid_{g_0X}$. Now we have

$$\begin{aligned} & \int_{Z_G(F)\backslash T(F)} c_f(t)D^H(t)\Delta(t)\omega(t)dt \\ = & \int_{T(F)\cap(g_0X)} c_f(t)D^H(t)\Delta(t)\omega(t)dt \\ = & \int_{T(F)\cap(g_0X)} c_{f'}(t)D^H(t)\Delta(t)\omega(\det(g_0))dt \\ = & \ \omega(\det(g_0))\int_{T(F)\cap(g_0X)} c_{f'}(t)D^H(t)\Delta(t)dt \\ = & \ \omega(\det(g_0))\int_{Z_G(F)\backslash T(F)} c_{f'}(t)D^H(t)\Delta(t)dt \end{aligned}$$

and

$$\begin{aligned} I(f,g) &= \int_{Z_H(F)\backslash H(F)} {}^g f^\xi(x)\omega(\det(x))dx \\ &= \int_{H(F)\cap(g_0X)} {}^g f^\xi(x)\omega(\det(x))dx \\ &= \int_{H(F)\cap(g_0X)} {}^g (f')^\xi(x)\omega(\det(g_0))dx \\ &= \omega(\det(g_0))\int_{H(F)\cap(g_0X)} {}^g (f')^\xi(x)dx \\ &= \omega(\det(g_0))\int_{Z_H(F)\backslash H(F)} {}^g (f')^\xi(x)dx \\ &= \omega(\det(g_0))I(f',g). \end{aligned}$$

This implies

$$I_{geom}(f) = \omega(\det(g_0))I_{geom}(f'),\ I_N(f) = \omega(\det(g_0))I_N(f'). \tag{5.9}$$

Since the geometric expansion holds for the function f', we know

$$\lim_{N\to\infty} I_N(f') = I(f').$$

Combining it with (5.9), we prove the geometric expansion for the function f, and this finishes the proof of the proposition. $\square$

5.3 The Lie Algebra Version of the Geometric Expansion

In this sectoin, we will talk about the Lie algebra analogy of the geometric side of the trace formula. This will be used in our proof of the group case. Let $f \in C_c^{\infty}(\mathfrak{g}_0(F))$ be a strongly cuspidal function. Define the function f^{ξ} on $\mathfrak{h}_0(F)$ by

$$f^{\xi}(Y) = \int_{\mathfrak{u}(F)} f(Y+N)\xi(N)dN.$$

For $g \in G(F)$, define

$$I(f,g) = \int_{\mathfrak{h}_0(F)} {}^{g}f^{\xi}(Y)dY,$$

and for each $N \in \mathbb{N}$, define

$$I_N(f) = \int_{U(F)H(F)\backslash G(F)} I(f,g)\kappa_N(g)dg. \tag{5.10}$$

As in Section 5.1, for each $T \in \mathcal{T}$, we can define the function $c_f = c_{\theta_f}$ on $\mathfrak{t}_{0,reg}(F)$, and define

$$I_{geom}(f) = \sum_{T\in\mathcal{T}} \mid W(H,T) \mid^{-1} \nu(T) \int_{\mathfrak{t}_0(F)} c_f(Y)D^H(Y)\Delta(Y)dY. \tag{5.11}$$

By a similar argument as Proposition 5.1.2, we know that the integral in (5.11) is absolutely convergent. The following theorem can be viewed as the Lie algebra version of the geometric expansion.

Theorem 5.3.1. *For every strongly cuspidal function $f \in C_c^{\infty}(\mathfrak{g}_0(F))$, we have*

$$\lim_{N\to\infty} I_N(f) = I_{geom}(f). \tag{5.12}$$

5.4 The Trace Formulas for the Reduced Models

In this section, we will talk about the trace formulas for the reduced models. The proofs of these trace formulas are the same as the Ginzburg-Rallis model case, so we will assume by induction that these trace formulas hold.

The distribution $I(f)$ in the trace formula is the same as the Ginzburg-Rallis model case we discussed in the previous sections. To be specific, we will still use $(G_{\bar{Q}}, R_{\bar{Q}})$ to denote the reduced models, and the character on $R_{\bar{Q}}(F)$ is just $(\omega\otimes\xi)|_{R_{\bar{Q}}(F)}\otimes\delta_{R_{\bar{Q}}}^{1/2}$. Let f be a strongly cuspidal function on $G_{\bar{Q}}(F)$ whose central character ω_f equals χ^{-2} on $Z_G(F)$, as in the Ginzburg-Rallis model case, for $g\in G_{\bar{Q}}(F)$, we define

$$I(f,g)=\int_{Z_G(F)\backslash R_{\bar{Q}}(F)} {}^g f(x)(\omega\otimes\xi)|_{R_{\bar{Q}}(F)}\otimes\delta_{R_{\bar{Q}}}^{1/2}(x)dx.$$

Then we define

$$I(f)=\int_{R_{\bar{Q}}(F)Z_{G_{\bar{Q}}}(F)\backslash G_{\bar{Q}}(F)} I(f,g)dg.$$

By a similar argument as in Appendix B, one can show that the integral above is absolutely convergent. $I(f)$ is the distribution in the trace formula. Same as the Ginzburg-Rallis case, when we prove the geometric side of the trace formula, we need to introduce some truncated function. We will skip the details here.

The spectral side of the trace formula is the same as the Ginzburg-Rallis model case. In other word, let

$$I_{spec}(f)=\int_{\Pi_{temp}(G_{\bar{Q}},\omega_f^{-1})}\theta_f(\tau)m(\bar{\tau})d\tau$$

where $m(\tau)$ is the multiplicity for the reduced model $(G_{\bar{Q}}, R_{\bar{Q}})$.

For the geometric side, it is more complicated. **We first discuss the trilinear GL_2 model case.** Let $\mathcal{T}$ be the subset of subtori of $H_{\bar{Q}}=H\cap\bar{Q}=H$ defined in Section 5.1. For $T=\{1\}$, we are in the split case, we still define $c_f(1)$ to be the germ of θ_f at the identity element, i.e. $c_f(1)=c_{\theta_f,\mathcal{O}_{reg}}(1)$. For $T=T_v$ with $1\neq v\in F^\times/(F^\times)^2$, and for $t\in T_v(F)_{reg}$, it is easy to see that $G_{\bar{Q}}(F)_t$ is just $(T_v(F))^3$, which is an abelian group. As a result, the germ expansion at t is just the quasi-character itself, so we define $c_f(t)=\theta_f(t)$. Finally, we define the geometric expansion to be

$$I_{geom}(f)=\sum_{T\in\mathcal{T}}|W(H,T)|^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)} c_f(t)D^H(t)\omega(h)dt.$$

Then we talk about the middle model case. Still we let $\mathcal{T}$ be as in Section 5.1. For $T=\{1\}$, we still let $c_f(1)=c_{\theta_f,\mathcal{O}_{reg}}(1)$. For $T=T_v(F)$ and $t\in T_v(F)_{reg}$, $G_{\bar{Q}}(F)_t$ is F-isomorphic to $\mathrm{GL}_2(F_v)\times\mathrm{GL}_1(F_v)$. Let $\mathcal{O}_v$ be the unique regular nilpotent orbit in $\mathfrak{gl}_2(F_v)\times\mathfrak{gl}_1(F_v)$ and we define $c_f(t)=c_{\theta_f,\mathcal{O}_v}(t)$. For $x\in H_{ss}(F)=H_{\bar{Q},ss}(F)$, we define

$$\Delta_Q(x)=|\det((1-ad(x)^{-1})|_{U_{\bar{Q}}(F)/U_{\bar{Q}}(F)_x})|_F.$$

Finally, we define the geometric expansion to be

$$I_{geom}(f)=\sum_{T\in\mathcal{T}}|W(H,T)|^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)}c_f(t)D^H(t)\Delta_Q(t)\omega(h)dt.$$

For Type II model, the geometric side is easy. To be specific, we define

$$I_{geom}(f)=c_f(1)$$

where $c_f(1)=c_{\theta_f,\mathcal{O}_{reg}}(1)$ is the germ of θ_f at the identity element. **As we mentioned in Section 4.5, the most important feature for Type II model is that every semisimple element in $R_{\bar{Q}}(F)$ is split. As a result, the only term in the geometric expansion is just the germ at the identity element.** Another way to explain this is that the only element in $\mathcal{T}\cap R_{\bar{Q}}$ is just the identity element.

Now we are ready to state our trace formula.

Theorem 5.4.1. *With the notations above, we have*

$$I_{geom}(f)=I(f)=I_{spec}(f).$$

As mentioned before, by induction, we will assume that Theorem 5.4.1 holds for all reduced models. Moreover, by the same argument as in Chapter 13, we can deduce a multiplicity formula for the reduced models from the trace formula. To be specific, for every irreducible tempered representation π of $G_{\bar{Q}}(F)$ whose central character equals χ^2 on $Z_G(F)$, we define $m_{geom}(\pi)$ as follows (similar to the definition of $I_{geom}(f)$):

- If we are in the trilinear GL_2 model case, define

  $$m_{geom}(\pi)=\sum_{T\in\mathcal{T}}|W(H,T)|^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)}c_\pi(t)D^H(t)\omega^{-1}(h)dt.$$

  Here $c_\pi(t)$ is defined in the same way as $c_f(t)$ except that we replace θ_f by θ_π.

- If we are in the middle model case, define

  $$m_{geom}(\pi)=\sum_{T\in\mathcal{T}}|W(H,T)|^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)}c_\pi(t)D^H(t)\Delta_Q(t)\omega^{-1}(h)dt.$$

  Here $c_\pi(t)$ is defined in the same way as $c_f(t)$ except that we replace θ_f by θ_π.

- If we are in the Type II reduced model case, let

$$m_{geom}(\pi) = c_\pi(1)$$

where $c_\pi(1) = c_{\theta_\pi, \mathcal{O}_{reg}}(1)$.

Then we can prove the following theorem for the reduced models.

Theorem 5.4.2. *With the notations above, we have*

$$m(\pi) = m_{geom}(\pi).$$

Remark 5.4.3. *By the theorem above, for Type II reduced models, the multiplicity is always equal to 1 for tempered representations.*

Chapter 6

Explicit Interwining Operator

In this chapter, we study an explicit element $\mathcal{L}_\pi$ in the Hom space given by the (normalized) integral of the matrix coefficients. The main result of this section is to show that the Hom space is nonzero if and only if $\mathcal{L}_\pi \neq 0$ (i.e. Theorem 6.2.1). In Sections 6.1 and 6.2, we define $\mathcal{L}_\pi$ and prove some basic properties of it. In Sections 6.3 and 6.4, we study the behavior of $\mathcal{L}_\pi$ under parabolic induction. Since we can not always reduce to the strongly tempered case, we have to treat the p-adic case and the archimedean case separately. In Section 6.5, we prove Theorem 6.2.1. Then in Section 5.6, we discuss some applications of Theorem 6.2.1, which are Corollary 6.6.2 and Corollary 6.6.4. These two results will play essential roles in our proof of the main results of this paper.

6.1 A Normalized Integral

Let χ be an unitary characters of $F^\times$, and let $\eta = \chi^2$. In Chapter 1, we define the character ω and ξ on $R(F)$. By Lemma 4.3.1, for all $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, the integral

$$\int_{Z_R(F)\backslash R(F)} f(h)\xi(h)\omega(h)dh$$

is absolutely convergent and defines a continuous linear form on the space $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$. By the next proposition, we can extend this linear form to the space $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$.

Proposition 6.1.1. *The linear form*

$$f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1}) \to \int_{Z_R(F)\backslash R(F)} f(h)\xi(h)\omega(h)dh$$

can be extended continuously to $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$.

Proof. Let $a:\mathbb{G}_m(F)\to Z_{G_0}(F)$ be a homomorphism defined by $a(t)=diag(t,t,1,1,t^{-1},t^{-1})$ in the split case, and $a(t)=diag(t,1,t^{-1})$ in the non-split case. Then we know that $\lambda(a(t)ha(t)^{-1})=t\lambda(h)$ for all $h\in R(F)$ and $t\in\mathbb{G}_m(F)$.

If F is p-adic, fix an open compact subgroup $K\subset G(F)$ (not necessarily maximal), it is enough to prove that the linear form

$$f\in\mathcal{C}_K(Z_G(F)\backslash G(F),\eta^{-1})\to\int_{Z_R(F)\backslash R(F)}f(h)\xi(h)\omega(h)dh$$

extends continuously to $\mathcal{C}_K^w(Z_G(F)\backslash G(F),\eta^{-1})$ for all K. Here we define $\mathcal{C}_K^w(Z_G(F)\backslash G(F),\eta^{-1})$ to be the space of bi-K-invariant elements in $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$. Let $K_a=a^{-1}(K\cap Z_{G_0}(F))$. It is an open compact subset of $F^\times$. Then for $f\in\mathcal{C}_K(Z_G(F)\backslash G(F),\eta^{-1})$, we have

$$\begin{aligned}
&\int_{Z_R(F)\backslash R(F)}f(h)\xi(h)\omega(h)dh\\
=\;& mes(K_a)^{-1}\int_{K_a}\int_{Z_R(F)\backslash R(F)}f(a(t)^{-1}ha(t))\xi(h)\omega(a(t)^{-1}ha(t))dhd^\times t\\
=\;& mes(K_a)^{-1}\int_{Z_R(F)\backslash R(F)}f(h)\omega(h)\int_{K_a}\xi(a(t)ha(t)^{-1})d^\times tdh\\
=\;& mes(K_a)^{-1}\int_{Z_R(F)\backslash R(F)}f(h)\omega(h)\int_{K_a}\psi(t\lambda(h))\mid t\mid^{-1}dtdh.
\end{aligned}$$

The function $x\in F\mapsto\int_{K_a}\psi(tx)\mid t\mid^{-1}dt$ is the Fourier transform of the function $\mid\cdot\mid^{-1}1_{K_a}\in C_c^\infty(F)$, so it also belongs to $C_c^\infty(F)$. Hence the last integral above is essentially bounded by

$$\int_{Z_R(F)\backslash R(F)}|f(h)|(1+\mid\lambda(h)\mid)^{-\delta}dh$$

for all $\delta>0$. Then by applying Lemma 4.3.1, we know that the integral above is also absolutely convergent for $f\in\mathcal{C}_K^w(Z_G(F)\backslash G(F),\eta^{-1})$. Thus the linear form can be extended continuously to $\mathcal{C}_K^w(Z_G(F)\backslash G(F),\eta^{-1})$.

If F=$\mathbb{R}$, recall that for $g\in G(F)$ and $f\in C^\infty(G(F))$, we have defined ${}^gf(x)=f(g^{-1}xg)$. Let Ad_a be a smooth representation of $F^\times$ on $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$ given by $Ad_a(t)(f)={}^{a(t)}f$. This induces an action of $\mathcal{U}(\mathfrak{gl}_1(F))$ on $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$, which is still denoted by Ad_a. Let $\Delta = 1-(t\frac{d}{dt})^2 \in \mathcal{U}(\mathfrak{gl}_1(F))$. By elliptic regularity (see Lemma 3.7 of [BK14]), for all integer $m \geq 1$, there exist $\varphi_1 \in C_c^{2m-2}(F^\times)$ and $\varphi_2 \in C_c^\infty(F^\times)$ such that $\varphi_1 * \Delta^m + \varphi_2 = \delta_1$. This implies

$$Ad_a(\varphi_1)Ad_a(\Delta^m) + Ad_a(\varphi_2) = Id.$$

Therefore for all $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, we have

$$\begin{aligned}
& \int_{Z_R(F)\backslash R(F)} f(h)\xi(h)\omega(h)dh \\
= & \int_{Z_R(F)\backslash R(F)} (Ad_a(\varphi_1)Ad_a(\Delta^m)f)(h)\xi(h)\omega(h)dh \\
& + \int_{Z_R(F)\backslash R(F)} (Ad_a(\varphi_2)f)(h)\xi(h)\omega(h)dh \\
= & \int_{Z_R(F)\backslash R(F)} (Ad_a(\Delta^m)f)(h)\omega(h) \int_{F^\times} \varphi_1(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
& + \int_{Z_R(F)\backslash R(F)} f(h)\omega(h) \int_{F^\times} \varphi_2(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
= & \int_{Z_R(F)\backslash R(F)} (Ad_a(\Delta^m)f)(h)\omega(h) \int_{F^\times} \varphi_1(t)\psi(t\lambda(h))\delta_P(a(t)) \mid t \mid^{-1} dtdh \\
& + \int_{Z_R(F)\backslash R(F)} f(h)\omega(h) \int_{F^\times} \varphi_2(t)\psi(t\lambda(h))\delta_P(a(t)) \mid t \mid^{-1} dtdh.
\end{aligned}$$

Here the second equation is to take the transform $h \mapsto a(t)^{-1}ha(t)$ in both integrals and the extra $\delta_P(a(t))$ is its Jacobian. For $i = 1, 2$, the functions $f_i : x \in F \to \int_F \varphi_i(t)\delta_P(a(t))|t|^{-1}\psi(tx)dt$ are the Fourier transforms of the functions $t \to \varphi_i(t)\delta_P(a(t))|t|^{-1} \in C_c^{2m-2}(F)$. Hence f_1 and f_2 are essentially bounded by $(1+|x|)^{-2m+2}$. By applying Lemma 4.3.1 again, we know that for all $m \geq 2$, the last two integrals above are absolutely convergent for all $f \in \mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$. Therefore the linear form can be extended continuously to $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$.

If $F = \mathbb{C}$, still let Ad_a be a smooth representation of $F^\times$ on $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$ given by $Ad_a(t)(f) = {}^{a(t)}f$. This induces an action of $\mathcal{U}(\mathfrak{gl}_1(\mathbb{C}))$ on $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$, which is still denoted by Ad_a. Fix a basis X_1, X_2 of $\mathfrak{gl}_1(\mathbb{C})$ as an $\mathbb{R}$-vector space, and let $\Delta_{\mathbb{C}} := 1 - X_1^2 - X_2^2 \in \mathcal{U}(\mathfrak{gl}_1(\mathbb{C}))$. By applying elliptic regularity in Lemma 3.7 of [BK14] again, for all integer $m \geq 2$, there exist $\varphi_1 \in C_c^{2m-3,\mathbb{R}}(\mathbb{C}^\times)$ and $\varphi_2 \in C_c^{\infty,\mathbb{R}}(\mathbb{C}^\times)$ such that

$$Ad_a(\varphi_1)Ad_a(\Delta_{\mathbb{C}}^m) + Ad_a(\varphi_2) = Id.$$

Here for any function $f \in C_c(\mathbb{C}^\times)$, we can view f as a function inside the space $C_c(\mathbb{R}^2)$. We then define the subspace $C_c^{2m-3,\mathbb{R}}(\mathbb{C}^\times)$ (resp. $C_c^{\infty,\mathbb{R}}(\mathbb{C}^\times)$) to be $C_c(\mathbb{C}^\times)\cap C_c^{2m-3}(\mathbb{R}^2)$ (resp. $C_c(\mathbb{C}^\times)\cap C_c^{\infty}(\mathbb{R}^2)$). Without loss of generality, we assume that the character ψ is defined to be $\psi(x)=\psi_0(Im(x))$ for some additive character ψ_0 on $\mathbb{R}$. Then for all $f \in \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})$, we have

$$\begin{aligned}
&\int_{Z_R(F)\backslash R(F)} f(h)\xi(h)\omega(h)dh \\
= &\int_{Z_R(F)\backslash R(F)} (Ad_a(\varphi_1)Ad_a(\Delta_{\mathbb{C}}^m)f)(h)\xi(h)\omega(h)dh \\
&+\int_{Z_R(F)\backslash R(F)} (Ad_a(\varphi_2)f)(h)\xi(h)\omega(h)dh \\
= &\int_{Z_R(F)\backslash R(F)} (Ad_a(\Delta_{\mathbb{C}}^m f)(h)\omega(h)\int_{\mathbb{C}^\times}\varphi_1(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times tdh \\
&+\int_{Z_R(F)\backslash R(F)} f(h)\omega(h)\int_{\mathbb{C}^\times}\varphi_2(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times tdh \\
= &\int_{Z_R(F)\backslash R(F)} (Ad_a(\Delta_{\mathbb{C}}^m)f)(h)\omega(h)\int_{\mathbb{C}^\times}\varphi_1(t)\psi_0(Re(t)Im(\lambda(h))+Im(t)Re(\lambda(h)))\delta_P(a(t))\mid t\mid^{-1}dtdh \\
&+\int_{Z_R(F)\backslash R(F)} f(h)\omega(h)\int_{\mathbb{C}^\times}\varphi_2(t)\psi_0(Re(t)Im(\lambda(h))+Im(t)Re(\lambda(h)))\delta_P(a(t))\mid t\mid^{-1}dtdh.
\end{aligned}$$

For $i=1,2$, the functions $f_i: x\in\mathbb{C}=\mathbb{R}^2\to\int_{\mathbb{C}}\varphi_i(t)\delta_P(a(t))|t|^{-1}\psi_0(Re(t)Im(x)+Im(t)Re(x))dt$ are the Fourier transforms of the functions $t\to\varphi_i(t)\delta_P(a(t))|t|^{-1}\psi(t\cdot x)\in C_c^{2m-3}(\mathbb{R}^2)$. Hence they are essentially bounded by $(1+|x|)^{-2m+3}$. By applying Lemma 4.3.1 again, we know that for all $m\geq 2$, the last two integrals above are absolutely convergent for all $f\in\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$. Therefore the linear form can be extended continuously to $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$. □

Denote by $\mathcal{P}_{R,\xi}$ the continuous linear form on $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$ defined above. i.e.

$$f\in\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})\to\int^*_{Z_R(F)\backslash R(F)} f(h)\xi(h)\omega(h)dh.$$

Lemma 6.1.2. *1. For all $f\in\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$, and $h_0,h_1\in R(F)$, we have*

$$\mathcal{P}_{R,\xi}(L(h_0)R(h_1)f)=\xi(h_0)\omega(h_0)\xi(h_1)^{-1}\omega(h_1)^{-1}\mathcal{P}_{R,\xi}(f)$$

where R (resp. L) is the right (resp. left) translation.

2. *Let $\varphi \in C_c^\infty(F^\times)$, and set $\varphi'(t) = \mid t \mid^{-1} \delta_P(a(t))\varphi(t)$. We can view both φ and φ' as elements in $C_c^\infty(F)$. Let $\hat{\varphi}'$ be the Fourier transform of φ' with respect to ψ. Then we have*

$$\mathcal{P}_{R,\xi}(Ad_a(\varphi)f) = \int_{Z_R(F)\backslash R(F)} f(h)\omega(h)\hat{\varphi}'(\lambda(h))dh$$

for all $f \in \mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$. Note that the last integral is absolutely convergent by Lemma 4.3.1

Proof. Since both sides of the equality are continuous in $\mathcal{C}^w(Z_G(F)\backslash G(F))$, it is enough to check (1) and (2) for $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$. In this case, $\mathcal{P}_{R,\xi}(f) = \int_{Z_R(F)\backslash R(F)} f(h)\xi(h)\omega(h)dh$. Then (1) follows from change variables in the integral. For (2), we have

$$\begin{aligned}
\mathcal{P}_{R,\xi}(Ad_a(\varphi)f) &= \int_{Z_R(F)\backslash R(F)} Ad_a(\varphi)(f)\xi(h)\omega(h)dh \\
&= \int_{Z_R(F)\backslash R(F)} f(h)\omega(h) \int_{F^\times} \varphi(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
&= \int_{Z_R(F)\backslash R(F)} f(h)\omega(h) \int_{F} \varphi(t)\psi(t\lambda(h))\delta_P(a(t))|t|^{-1} dt dh \\
&= \int_{Z_R(F)\backslash R(F)} f(h)\omega(h)\hat{\varphi}'(\lambda(h))dh.
\end{aligned}$$

This finishes the proof of the Lemma. □

6.2 The Definition and Properties of $\mathcal{L}_\pi$

Let π be a tempered representation of $G(F)$ with central character η. For all $T \in \mathrm{End}(\pi)^\infty$, define

$$\mathcal{L}_\pi(T) = \mathcal{P}_{R,\xi}(\mathrm{tr}(\pi(g^{-1})T)) = \int^*_{Z_R(F)\backslash R(F)} \mathrm{tr}(\pi(h^{-1})T)\xi(h)\omega(h)dh.$$

By Proposition 6.1.1, together with the fact that the map $T \in \mathrm{End}(\pi)^\infty \to (g \to \mathrm{tr}(\pi(g^{-1})T) \in \mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$ is continuous, we know that $\mathcal{L}_\pi : \mathrm{End}(\pi)^\infty \to \mathbb{C}$ is a continuous linear form. By Lemma 6.1.2, for any $h, h' \in R(F)$, we have

$$\mathcal{L}_\pi(\pi(h)T\pi(h')) = \xi(hh')\omega(hh')\mathcal{L}_\pi(T). \tag{6.1}$$

For $e \in \pi^{\infty}, e' \in \bar{\pi}^{\infty}$, define $T_{e,e'} \in \mathrm{End}(\pi)^{\infty}$ to be $e_0 \in \pi \mapsto (e_0, e')e$. Set $\mathcal{L}_{\pi}(e,e') = \mathcal{L}_{\pi}(T_{e,e'})$. Then we have

$$\mathcal{L}_{\pi}(e,e') = \int_{Z_R(F)\backslash R(F)}^{*} (e, \pi(h)e')\omega(h)\xi(h)dh.$$

If we fix e', by (6.1), the map $e \in \pi^{\infty} \to \mathcal{L}_{\pi}(e,e')$ belongs to $Hom_H(\pi^{\infty}, \omega \otimes \xi)$. Since $Span\{T_{e,e'} \mid e \in \pi^{\infty}, e' \in \bar{\pi}^{\infty}\}$ is dense in $\mathrm{End}(\pi)^{\infty}$ (in p-adic case, they are equal), we have that $\mathcal{L}_{\pi} \neq 0 \Rightarrow m(\pi) \neq 0$. The purpose of this section is to prove the other direction.

Theorem 6.2.1. *For all $\pi \in \Pi_{temp}(G,\eta)$, we have*

$$\mathcal{L}_{\pi} \neq 0 \iff m(\pi) \neq 0.$$

Our proof for this result is based on the method developed by Waldspurger ([W12, Proposition 5.7]) and by Beuzart-Plessis ([B15, Theorem 8.2.1]) for the GGP models. See also [SV, Theorem 6.2.1]. The key ingredient in the proof is the Plancherel formula, together with the fact that the nonvanishing property of $\mathcal{L}_{\pi}$ is invariant under the parabolic induction and the unramified twist. For the rest of this section, we discuss some basic properties of $\mathcal{L}_{\pi}$.

The operator $\mathcal{L}_{\pi}$ defines a continuous linear map

$$L_{\pi} : \pi^{\infty} \to \bar{\pi}^{-\infty},\ e \to \mathcal{L}_{\pi}(e, \cdot)$$

where $\bar{\pi}^{-\infty}$ is the topological dual of π^{∞} endowed with the strong topology. The image of L_{π} belongs to $(\bar{\pi}^{-\infty})^{R,\omega\otimes\xi} = Hom_R(\pi^{\infty}, \omega \otimes \xi)$. So if π is irreducible, the image is of dimension less or equal to 1. Let $T \in \mathrm{End}(\pi)^{\infty}$. It can be uniquely extended to a continuous operator $T : \bar{\pi}^{-\infty} \to \pi^{\infty}$. Then we have the following two operators, which are both of finite rank:

$$TL_{\pi} : \pi^{\infty} \to \pi^{\infty},\ L_{\pi}T : \bar{\pi}^{-\infty} \to \bar{\pi}^{-\infty}.$$

In particular, they are of trace class. It is easy to see that

$$\mathrm{tr}(TL_{\pi}) = \mathrm{tr}(L_{\pi}T) = \mathcal{L}_{\pi}(T). \tag{6.2}$$

Lemma 6.2.2. *With the notation above, the followings hold.*

1. *The map* $\pi \in \Pi_{temp}(G,\eta) \to L_\pi \in Hom(\pi^\infty, \bar{\pi}^{-\infty})$ *is smooth in the following sense: For all parabolic subgroup* $Q = LU_Q$ *of* G, $\sigma \in \Pi_2(L)$, *and for all maximal compact subgroup* K *of* $G(F)$, *the map* $\lambda \in i\mathfrak{a}_{L,0}^* \to \mathcal{L}_{\pi_\lambda} \in \mathrm{End}(\pi_\lambda)^{-\infty} \simeq \mathrm{End}(\pi_K)^{-\infty}$ *is smooth, here* $\pi_\lambda = I_Q^G(\sigma_\lambda)$ *and* $\pi_K = I_{Q\cap K}^K(\sigma_K)$.

2. *For* $\pi \in \Pi_{temp}(G,\eta)$, *and for all* $S,T \in \mathrm{End}(\pi)^\infty$, *we have* $SL_\pi T \in \mathrm{End}(\pi)^\infty$, *and* $\mathcal{L}_\pi(S)\mathcal{L}_\pi(T) = \mathcal{L}_\pi(SL_\pi T)$.

3. *For* $S,T \in \mathcal{C}(\Pi_{temp}(G,\eta))$, *the section* $\pi \in \Pi_{temp}(G,\eta) \mapsto S_\pi L_\pi T_\pi \in \mathrm{End}(\pi)^\infty$ *belongs to* $C^\infty(\Pi_{temp}(G,\eta))$.

4. *For* $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, *assume that its Fourier transform* $\pi \in \Pi_{temp}(G,\eta) \to \pi(f)$ *is compactly supported (this is always true in p-adic case). Then we have*
$$\int_{Z_R(F)\backslash R(F)} f(h)\xi(h)\omega(h)dh = \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(f))\mu(\pi)d\pi$$
with both integrals being absolutely convergent.

5. *For* $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ *and* $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$, *assume that the Fourier transform of* f *is compactly supported. Then we have*
$$\begin{aligned} &\int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(f))\overline{\mathcal{L}_\pi(\pi(\bar{f}'))}\mu(\pi)d\pi \\ = &\int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \int_{Z_G(F)\backslash G(F)} f(hgh')f'(g)dg\xi(h')\omega(h')dh'\xi(h)\omega(h)dh \end{aligned}$$
where the left hand side is absolutely convergent and the right hand side is convergent in that order but is not necessarily absolutely convergent.

Proof. (1), (2) and (3) follow from the same argument as Lemma 8.2.1 of [B15], we will skip it here. The proof of (4) and (5) is also similar to the loc. cit. (except that we need to take care of the center of the group), we only include the proof here for completion.

For (4), by Lemma 4.3.1, the left hand side is absolutely convergent. Since the Fourier transform of f is compactly supported, the right hand side is also absolutely convergent. Let $\varphi(f,\pi)(g) = \mathrm{tr}(\pi(g^{-1})\pi(f))$, which is a function in $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$. By the Plancherel formula in Section 2.8, we have
$$f = \int_{\Pi_{temp}(G,\eta)} \varphi(f,\pi)\mu(\pi)d\pi.$$

By applying the operator $\mathcal{P}_{R,\xi}$ on both sides, we have

$$\mathcal{P}_{R,\xi}(f) = \int_{\Pi_{temp}(G,\eta)} \mathcal{P}_{R,\xi}(\varphi(f,\pi))\mu(\pi)d\pi.$$

This proves (4).

For (5), let $f'^{\vee}(g) = f'(g^{-1})$. Then the right hand side is equal to

$$\int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} (f'^{\vee} * L(h^{-1})f)(h')\xi(h')\omega(h')dh'\xi(h)\omega(h)dh. \tag{6.3}$$

The Fourier transform of f is compactly supported, so is $f'^{\vee} * L(h^{-1})f$. By applying part (4) to $f'^{\vee} * L(h^{-1})f$, we know that the inner integral in (6.3) is absolutely convergent and we have

$$\begin{aligned} & \int_{Z_R(F)\backslash R(F)} (f'^{\vee} * L(h^{-1})f)(h')\xi(h')\omega(h')dh' \\ = & \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_{\pi}(\pi(f'^{\vee})\pi(h^{-1})\pi(f))\mu(\pi)d\pi \\ = & \int_{\Pi_{temp}(G,\eta)} \mathrm{tr}(\pi(h^{-1})\pi(f)L_{\pi}\pi(f'^{\vee}))\mu(\pi)d\pi. \end{aligned}$$

The last equality holds because of (6.2). By part (3), the section $\pi \in \Pi_{temp}(G,\eta) \mapsto \pi(f)L_{\pi}\pi(f'^{\vee})$ is smooth, and is also compactly supported, and hence it belongs to $\mathcal{C}(\Pi_{temp}(G,\eta))$. By the matrical Paley-Wiener Theorem in Section 2.8, it is a Fourier transform of a Harish-Chandra-Schwartz function. Applying part (4) to such a function, we know the exterior integral of (6.3) is absolutely convergent and the whole expression is equal to

$$\int_{\Pi_{temp}(G,\eta)} \mathcal{L}_{\pi}(\pi(f)L_{\pi}\pi(f'^{\vee}))\mu(\pi)d\pi.$$

By part (2) and the fact that $\mathcal{L}_{\pi}(\pi(f'^{\vee})) = \overline{\mathcal{L}_{\pi}(\pi(\bar{f}'))}$, (6.3) is then equal to

$$\int_{\Pi_{temp}(G,\eta)} \mathcal{L}_{\pi}(\pi(f))\overline{\mathcal{L}_{\pi}(\pi(\bar{f}'))}\mu(\pi)d\pi.$$

This finishes the proof of the lemma. $\square$

The next lemma is about the asymptotic properties for elements in $Hom_R(\pi,\omega\otimes\xi)$.

Lemma 6.2.3. *Assume that $F \neq \mathbb{C}$, the followings hold.*

1. *Let π be a tempered representation of $G(F)$ with central character η and $l \in Hom_R(\pi, \omega \otimes \xi)$ be a continuous $(R, \omega \otimes \xi)$-equivariant linear form. Then there exist $d > 0$ and a continuous semi-norm ν_d on π such that*

$$|l(\pi(x)e)| \leq \nu_d(e)\Xi^{R\backslash G}(x)\sigma_{R\backslash G}(x)^d$$

for all $e \in \pi$ and $x \in R(F)\backslash G(F)$.

2. *For all $d > 0$, there exist $d' > 0$ and a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}_d^w(Z_G(F)\backslash G(F), \eta^{-1})$ such that*

$$|\mathcal{P}_{R,\xi}(R(x)L(y)\varphi)| \leq \nu_{d,d'}(\varphi)\Xi^{R\backslash G}(x)\Xi^{R\backslash G}(y)\sigma_{R\backslash G}(x)^d\sigma_{R\backslash G}(y)^{d'}$$

for all $\varphi \in \mathcal{C}_d^w(Z_G(F)\backslash G(F), \eta^{-1})$ and $x, y \in R(F)\backslash G(F)$.

Proof. The proof is similar to the GGP case as in Lemma 8.3.1 of [B15], we only include it here for completion. We use the same notation as in Chapter 4. In other words,

- $\bar{P}_0 = M_0\bar{U}_0$ is a good minimal parabolic subgroup of G_0, $A_0 = A_{M_0}$.
- $A_0^+ = \{a_0 \in A_0(F) \mid\mid \alpha(a_0) \mid\geq 1 \; for \; all \; \alpha \in \Psi(A_0, \bar{P}_0)\}$.
- $\bar{P}_{min} = \bar{P}_0\bar{U} = M_{min}\bar{U}_{min}$ is a good minimal parabolic subgroup of G, $A_{min} = A_{M_{min}} = A_0$.
- Δ is the set of simple roots of A_{min} in P_{min}, and $\Delta_P = \Delta \cap \Psi(A_{min}, P)$.

We first prove part 1. By the weak Cartan decomposition in Proposition 4.2.3, together with Proposition 4.4.1(1) and (2), it is enough to show that there exists a continuous semi-norm ν on π^∞ such that

$$|l(\pi(a)e)| \leq \Xi^G(a)\nu(e) \tag{6.4}$$

for all $e \in \pi^\infty$ and $a \in A_0^+$.

If F is p-adic, the topology on π^∞ is the finest locally convex topology. We only need to show that for all $e \in \pi^\infty$, we have

$$|l(\pi(a)e)| \ll \Xi^G(a) \tag{6.5}$$

for all $a \in A_0^+$. For $e \in \pi^\infty$, choose an open compact subgroup $K \subset G(F)$ such that e is an K-fixed vector. We first prove the following claim.

Claim 6.2.4. *There exists $c = c_K \geq 1$ such that for all $a \in A_{min}(F)$, if there exists $\alpha \in \Delta_P$ such that $|\alpha(a)| \geq c$, then*

$$l(\pi(a)e) = 0.$$

In fact, let $\alpha \in \Delta_P$ and let $a \in A_{min}(F)$. By Proposition 4.2.3(3), there exists $X \in \mathfrak{n}_\alpha(F)$ such that $\xi(e^X) \neq 1$. Then if $|\alpha(a)|$ is large enough, we have $a^{-1}e^X a \in K$. This implies

$$\xi(e^X)l(\pi(a)e) = l(\pi(e^X)\pi(a)e) = l(\pi(a)e).$$

Therefore $l(\pi(a)e) = 0$, and this proves the claim.

Choose $c \geq 1$ as in the claim above, set

$$A_{min}^+(c) = \{a \in A_{min}(F) | |\alpha(a)| \leq c, \ \forall \alpha \in \Delta\}.$$

By the claim above, we only need to prove (6.5) for $a \in A_{min}^+(c)$. It is easy to see that there exists an open compact subgroup $K'_{\bar{P}_{min}}$ of $\bar{P}_{min}(F)$ such that

$$K'_{\bar{P}_{min}} \subset aKa^{-1} \cap \bar{P}_{min}$$

for all $a \in A_{min}^+(c)$. Let K'_R be an open compact subgroup of $R(F)$ such that $\omega \otimes \xi$ is trivial on it. Finally, choose an open compact subgroup $K' \subset G(F)$ such that $K' \subset K'_R K'_{\bar{P}_{min}}$. This is possible since $\bar{P}_{min}$ is a good parabolic subgroup. For $k' = k'_R k'_{\bar{P}_{min}} \in K'$ with $k'_R \in K'_R$ and $k'_{\bar{P}_{min}} \in K'_{\bar{P}_{min}}$, we have

$$l(\pi(k')\pi(a)e) = l(\pi(k'_R)\pi(a)\pi(a^{-1}k'_{\bar{P}_{min}}a)e) = \omega \otimes \xi(k'_R)l(\pi(a)e) = l(\pi(a)e)$$

for all $a \in A_{min}^+(c)$. Therefore

$$l(\pi(a)e) = l(\pi(e_{K'})\pi(a)e)$$

for all $a \in A_{min}^+(c)$. Here $e_{K'}$ is the characteristic function on K' multiply by $meas(K')^{-1}$. Since π is tempered,

$$|(\pi(g)e, e')| \ll \Xi^G(g)$$

for all $g \in G(F), \ e, e' \in \pi^\infty$. Together with the fact that $l \circ \pi(e_{K'}) \in \overline{\pi^\infty}$, we have

$$|l(\pi(a)e)| = |l(\pi(e_{K'})\pi(a)e)| \ll \Xi^G(a)$$

for all $a \in A_0^+$. This proves (6.5).

If $F = \mathbb{R}$. For $I \subset \Delta$, set

$$A_{min}^+(I) = \{a \in A_{min}(F) | |\alpha(a)| \leq 1 \ \forall \alpha \in \Delta \backslash I, \ |\alpha(a)| > 1 \ \forall \alpha \in I\}.$$

Then we have $A_0^+ = \cup_{I \subset \Delta_P} A_{min}^+(I)$. Therefore it is enough to prove (6.4) for $a \in A_{min}^+(I)$. Let $X_1, \cdots, X_p$ be a basis of $\bar{\mathfrak{p}}_{min}(F)$, and let k be an integer larger than $dim(\bar{P}_{min}) + 1$. Set

$$\Delta_{min} = 1 - (X_1^2 + \cdots + X_p^2) \in \mathcal{U}(\bar{\mathfrak{p}}_{min}).$$

The following claim is an easy consequence of Proposition 4.2.3(3).

Claim 6.2.5. *There exists $u = u_{I,k} \in \mathcal{U}(\mathfrak{u})$ such that the two maps*

$$a \in A_{min}^+(I) \mapsto a^{-1}(\Delta_{min}^k u)a \in \mathcal{U}(\mathfrak{g})$$

$$a \in A_{min}^+(I) \mapsto a^{-1}ua \in \mathcal{U}(\mathfrak{g})$$

have bounded images and $d(\omega \otimes \xi)(u) = 1$.

Fix $u \in \mathcal{U}(\mathfrak{u})$ as in the claim, by elliptic regularity (see Lemma 3.7 of [BK14]), we can find two functions $\varphi_1 \in C_c^{k_1}(\bar{P}_{min}(F))$ and $\varphi_2 \in C_c^{\infty}(\bar{P}_{min}(F))$ with $k_1 = 2k - dim(\bar{P}_{min}) - 1$, such that

$$\pi(\varphi_1)\pi(\Delta_{min}^k) + \pi(\varphi_2) = Id.$$

Choose $\varphi_R \in C_c^{\infty}(R(F))$ such that $\int_{R(F)} \varphi_R(h)\omega \otimes \xi(h)dh = 1$. Then for all $e \in \pi^{\infty}$ and $a \in A_{min}^+(I)$, we have

$$\begin{aligned}
l(\pi(a)e) &= d(\omega \otimes \xi)(u)l(\pi(a)e) = l(\pi(u)\pi(a)e) \\
&= l(\pi(\varphi_1)\pi(\Delta_{min}^k u)\pi(a)e) + l(\pi(\varphi_2)\pi(u)\pi(a)e) \\
&= l(\pi(\varphi_1)\pi(a)\pi(a^{-1}(\Delta_{min}^k u)a)e) + l(\pi(\varphi_2)\pi(a)\pi(a^{-1}ua)e) \\
&= l(\pi(\varphi_H * \varphi_1)\pi(a)\pi(a^{-1}(\Delta_{min}^k u)a)e) + l(\pi(\varphi_H * \varphi_2)\pi(a)\pi(a^{-1}ua)e).
\end{aligned}$$

Note that the functions $\varphi_H * \varphi_1$ and $\varphi_H * \varphi_2$ both belong to $C_c^{k_1}(G(F))$. Then once we let k large, there exists a continuous semi-norm ν on π^{∞} such that the last line of the equation above is bounded by

$$(\nu(\pi(a^{-1}(\Delta_{min}^k u)a)e) + \nu(\pi(a^{-1}ua)e))\Xi^G(a). \tag{6.6}$$

Then by the claim above, (6.6) is bounded by

$$\nu(e)\Xi^G(a).$$

This proves (6.4).

We then prove the second part. By the same reduction as in (1), we only need to show that there exists a continuous semi-norm ν_d on $\mathcal{C}_d^w(Z_G(F)\backslash G(F),\eta^{-1})$ such that

$$|\mathcal{P}_{R,\xi}(R(a_1)L(a_2)\varphi)| \leq \nu_d(\varphi)\Xi^G(a_1)\Xi^G(a_2)\sigma_0(a_1)^d\sigma_0(a_2)^d \tag{6.7}$$

for all $\varphi \in \mathcal{C}_d^w(Z_G(F)\backslash G(F),\eta^{-1})$ and $a_1,a_2 \in A_0^+$.

If F **is p-adic**, we fix an open compact subgroup $K \subset G(F)$. We only need to show that there exists a continuous semi-norm $\nu_{K,d}$ on $\mathcal{C}_{K,d}^w(Z_G(F)\backslash G(F),\eta^{-1})$ such that

$$|\mathcal{P}_{R,\xi}(R(a_1)L(a_2)\varphi)| \leq \nu_{K,d}(\varphi)\Xi^G(a_1)\Xi^G(a_2)\sigma_0(a_1)^d\sigma_0(a_2)^d \tag{6.8}$$

for all $\varphi \in \mathcal{C}_{K,d}^w(Z_G(F)\backslash G(F),\eta^{-1})$ and $a_1,a_2 \in A_0^+$. Then as in the proof of (1), we can find a constant $c = c_K \geq 1$ such that

$$\mathcal{P}_{R,\xi}(R(a_1)L(a_2)\varphi) = 0$$

for all $\varphi \in \mathcal{C}_{K,d}^w(Z_G(F)\backslash G(F),\eta^{-1})$ and $a_i \in A_0^+ - A_{min}^+(c)$ for some $i \in \{1,2\}$. Then by the same argument as in (1), we can find an open compact subgroup $K' \subset G(F)$ such that

$$\mathcal{P}_{R,\xi}(R(a_1)L(a_2)\varphi) = \mathcal{P}_{R,\xi}(R(e_{K'})L(e_{K'})R(a_1)L(a_2)\varphi)$$

for all $\varphi \in \mathcal{C}_{K,d}^w(Z_G(F)\backslash G(F),\eta^{-1})$ and $a_1,a_2 \in A_{min}^+(c)$. Finally (6.8) follows from Lemma 1.5.1(1) of [B15].

If $F = \mathbb{R}$, as in the proof of (1), we only need need to prove that for fixed $I,J \subset \Delta_P$, there exists a continuous semi-norm $\nu_{I,J,d}$ on $\mathcal{C}_d^w(Z_G(F)\backslash G(F),\eta^{-1})$ such that

$$|\mathcal{P}_{R,\xi}(R(a_1)L(a_2)\varphi)| \leq \nu_{I,J,d}(\varphi)\Xi^G(a_1)\Xi^G(a_2)\sigma_0(a_1)^d\sigma_0(a_2)^d \tag{6.9}$$

for all $\varphi \in \mathcal{C}_d^w(Z_G(F)\backslash G(F),\eta^{-1})$, $a_1 \in A_{min}^+(I)$ and $a_2 \in A_{min}^+(J)$.

Choose k,u_I,u_J as in the proof of (1). Then by the same argument, we can show that there exist functions $\varphi_1,\varphi_2,\varphi_3,\varphi_4 \in C_c^{k_1}(G(F))$ with $k_1 = 2k - dim(\bar{P}_{min}) - 1$, such that

$$\begin{aligned}\mathcal{P}_{R,\xi}(R(a_1)L(a_2)\varphi) &= \mathcal{P}_{R,\xi}(R(\varphi_1)L(\varphi_3)R(a_1)L(a_2)R(a_1^{-1}(\Delta_{min}^k u_I)a_1)L(a_2^{-1}(\Delta_{min}^k u_J)a_2)\varphi)\\ &+\mathcal{P}_{R,\xi}(R(\varphi_1)L(\varphi_4)R(a_1)L(a_2)R(a_1^{-1}(\Delta_{min}^k u_I)a_1)L(a_2^{-1}u_J a_2)\varphi)\\ &+\mathcal{P}_{R,\xi}(R(\varphi_2)L(\varphi_3)R(a_1)L(a_2)R(a_1^{-1}u_I a_1)L(a_2^{-1}(\Delta_{min}^k u_J)a_2)\varphi)\\ &+\mathcal{P}_{R,\xi}(R(\varphi_2)L(\varphi_4)R(a_1)L(a_2)R(a_1^{-1}u_I a_1)L(a_2^{-1}u_J a_2)\varphi)\end{aligned}$$

for all $\varphi \in \mathcal{C}_d^w(Z_G(F)\backslash G(F),\eta^{-1})$, $a_1 \in A_{min}^+(I)$ and $a_2 \in A_{min}^+(J)$. Then (6.9) follows from Lemma 1.5.1(1) of [B15] together with the fact that $a_1^{-1}u_I a_1$, $a_1^{-1}(\Delta_{min}^k u_I)a_1$, $a_2^{-1}u_J a_2$, $a_2^{-1}(\Delta_{min}^k u_J)a_2$ have bounded images. $\square$

6.3 Parabolic Induction for the p-adic Case

Assume that F is p-adic in this section. Let π be a tempered representation of $G(F)$ with central character η. There exists a parabolic subgroup $\bar{Q} = LU_{\bar{Q}}$ of G, together with a discrete series $\tau \in \Pi_2(L)$ such that $\pi = I_{\bar{Q}}^G(\tau)$. By Proposition 4.2.1, we may assume that $\bar{Q}$ is a good parabolic subgroup. We can further assume that the inner product on π is given by

$$(e,e') = \int_{Q(F)\backslash G(F)} (e(g),e'(g))_\tau dg,\ \forall e,e' \in \pi = I_Q^G(\tau). \tag{6.10}$$

Let $R_{\bar{Q}} = R \cap \bar{Q}$. For $T \in \mathrm{End}(\tau)^\infty$, define

$$\mathcal{L}_\tau(T_\tau) = \int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \mathrm{tr}(\tau(h_{\bar{Q}}^{-1})T)\delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}\omega(h_{\bar{Q}})\xi(h_{\bar{Q}})dh_{\bar{Q}}.$$

The integral above is absolutely convergent by Proposition 4.4.2(2) together with the assumption that τ is a discrete series. The purpose of the section is to prove the following proposition.

Proposition 6.3.1. *With the notation above, we have*

$$\mathcal{L}_\pi \neq 0 \iff \mathcal{L}_\tau \neq 0.$$

Proof. For $e,e' \in \pi^\infty$, by (6.10), we have

$$\mathcal{L}_\pi(e,e') = \int_{Z_R(F)\backslash R(F)}^* \int_{\bar{Q}(F)\backslash G(F)} (e(g),e'(gh))_\tau dg\omega(h)\xi(h)dh.$$

Same as in previous sections, let $a : \mathbb{G}_m(F) \to Z_{G_0}(F)$ be a homomorphism defined by $a(t) = diag(t, t, 1, 1, t^{-1}, t^{-1})$ in the split case, and $a(t) = diag(t, 1, t^{-1})$ in the non-split case. Since $e, e' \in \pi^\infty$, there exists an open compact subgroup K_0 of $G(F)$ such that the functions $e, e' : G(F) \to \tau$ is bi-K_0-invariant. Let $K_a = a^{-1}(K_0 \cap Z_{G_0}(F)) \subset F^\times$, which is an open compact subset. By Proposition 6.1.1, we have

$$\begin{aligned}\mathcal{L}_\pi(e, e') &= \int^*_{Z_R(F)\backslash R(F)} \int_{\bar{Q}(F)\backslash G(F)} (e(g), e'(gh))_\tau dg \xi(h)\omega(h)dh \\ &= meas(K_a)^{-1} \int_{Z_R(F)\backslash R(F)} \int_{\bar{Q}(F)\backslash G(F)} (e(g), e'(gh))_\tau dg \\ &\quad \times \int_{K_a} \psi(t\lambda(h)) \mid t \mid^{-1} dt\omega(h)dh.\end{aligned} \tag{6.11}$$

By the same proposition, the last two integrals $\int_{Z_R(F)\backslash R(F)} \int_{\bar{Q}(F)\backslash G(F)}$ above is absolutely convergent. Since $\bar{Q}$ is a good parabolic subgroup, by Proposition 4.2.1, we can choose the Haar measures compatibly so that for all $\varphi \in L_1(\bar{Q}(F)\backslash G(F), \delta_{\bar{Q}})$, we have

$$\int_{\bar{Q}(F)\backslash G(F)} \varphi(g)dg = \int_{R_{\bar{Q}}(F)\backslash R(F)} \varphi(h)dh.$$

Then (6.11) becomes

$$\begin{aligned}\mathcal{L}_\pi(e, e') &= meas(K_a)^{-1} \int_{Z_R(F)\backslash R(F)} \int_{R_{\bar{Q}}(F)\backslash R(F)} (e(h'), e'(h'h))_\tau dh' \\ &\quad \times \int_{K_a} \psi(t\lambda(h)) \mid t \mid^{-1} dt\omega(h)dh.\end{aligned}$$

The integral $\int_{Z_R(F)\backslash R(F)} \int_{R_{\bar{Q}}(F)\backslash R(F)}$ above is absolutely convergent because (6.11) is absolutely convergent. By switching the two integrals, making the transform $h \to h'h$ and decomposing $\int_{Z_R(F)\backslash R(F)}$ as $\int_{R_{\bar{Q}}(F)\backslash R(F)} \int_{Z_R(F)\backslash R_{\bar{Q}}(F)}$, we have

$$\mathcal{L}_\pi(e, e') = meas(K_a)^{-1} \int_{(R_{\bar{Q}}(F)\backslash R(F))^2} f(h, h')dhdh'$$

where

$$\begin{aligned}
f(h,h') &= \int_{Z_R(F)\backslash R_{\bar{Q}}(F)} (e(h), e'(h_{\bar{Q}}h'))_\tau \omega(h_{\bar{Q}})\omega(h^{-1}h') \\
&\quad \times \int_{K_a} \psi(t\lambda(h'))\psi(t\lambda(h_{\bar{Q}}))\psi(-t\lambda(h)) \mid t \mid^{-1} dt dh_{\bar{Q}} \qquad (6.12) \\
&= \int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h), \tau(h_{\bar{Q}})e'(h'))_\tau \omega(h_{\bar{Q}})\omega(h^{-1}h') \\
&\quad \times \int_{K_a} \psi(t\lambda(h'))\psi(t\lambda(h_{\bar{Q}}))\psi(-t\lambda(h)) \mid t \mid^{-1} dt dh_{\bar{Q}}.
\end{aligned}$$

Here we use the equation $\delta_{R_{\bar{Q}}}(h_{\bar{Q}}) = \delta_{\bar{Q}}(h_{\bar{Q}})$ in the second equality. We first show that the integral (6.12) is absolutely convergent for any $h, h' \in R_{\bar{Q}}(F)\backslash R(F)$. In fact, since K_a is compact, it is enough to show that for any $h, h' \in R_{\bar{Q}}(F)\backslash R(F)$, the integral

$$\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h), \tau(h_{\bar{Q}})e'(h'))_\tau dh_{\bar{Q}}$$

is absolutely convergent. This just follows from Proposition 4.4.2(2) together with the assumption that τ is discrete series. Then by switching the two integrals in (6.12), we have

$$\begin{aligned}
f(h,h') &= \int_{K_a}\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h), \tau(h_{\bar{Q}})e'(h'))_\tau \omega(h_{\bar{Q}}) \\
&\quad \times \psi(t\lambda(h_{\bar{Q}}))dh_{\bar{Q}}\omega(h^{-1}h')\psi(t\lambda(h'))\psi(-t\lambda(h)) \mid t \mid^{-1} dt.
\end{aligned}$$

By changing the variable $h_{\bar{Q}} \to a(t)h_{\bar{Q}}a(t)^{-1}$ in the inner integral (note that the Jacobian of such transform is 1 since $a(t) \in K_0$), we have

$$\begin{aligned}
&\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h), \tau(h_{\bar{Q}})e'(h'))_\tau \omega(h_{\bar{Q}})\psi(t\lambda(h_{\bar{Q}}))dh_{\bar{Q}} \\
= &\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h), \tau(a(t)^{-1}h_{\bar{Q}}a(t))e'(h'))_\tau \omega(h_{\bar{Q}})\psi(\lambda(h_{\bar{Q}}))dh_{\bar{Q}} \\
= &\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \delta_{R_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h), \tau(h_{\bar{Q}})e'(h'))_\tau \omega(h_{\bar{Q}})\psi(\lambda(h_{\bar{Q}}))dh_{\bar{Q}} \\
= &\ \mathcal{L}_\tau(e(h), e'(h')).
\end{aligned}$$

Here we use the fact that e' is bi-K_0-invariant. Then we have

$$f(h,h') = \int_{K_a} \mathcal{L}_\tau(e(h), e(h'))\omega(h^{-1}h')\psi(t\lambda(h'))\psi(-t\lambda(h)) \mid t \mid^{-1} dt.$$

If $\mathcal{L}_\pi(e,e') \neq 0$, there exist $h,h' \in R_{\bar{Q}}(F)\backslash R(F)$ such that $f(h,h') \neq 0$, and hence $\mathcal{L}_\tau(e(h),e(h')) \neq 0$. This proves that $\mathcal{L}_\pi \neq 0 \Rightarrow \mathcal{L}_\tau \neq 0$.

For the other direction, if $\mathcal{L}_\tau \neq 0$, we can find $v_0, v_0' \in \tau^\infty$ such that $\mathcal{L}_\tau(v_0,v_0') \neq 0$. We choose a small open subset $\mathcal{U} \subset R_{\bar{Q}}(F)\backslash R(F)$ and let $s : \mathcal{U} \to R(F)$ be an analytic section of the map $R(F) \to R_{\bar{Q}}(F)\backslash H(F)$. For $f,f' \in C_c^\infty(\mathcal{U})$, define $\varphi,\varphi' \in C_c^\infty(\mathcal{U},\tau^\infty)$ to be $\varphi(h) = f(h)v_0, \varphi'(h) = f'(h)v_0'$, then set

$$e_\varphi(g) = \begin{cases} \delta_{\bar{Q}}^{1/2}(l)\tau(l)\varphi(h) & \text{if} \quad g = lus(h) \; with \; l \in L(F), u \in U_{\bar{Q}}(F), h \in \mathcal{U}; \\ 0 & \text{else.} \end{cases}$$

This is an element of π^∞. Similarly we can define $e_{\varphi'}$. By the above discussion, we have

$$\mathcal{L}_\pi(e_\varphi, e_{\varphi'}) = meas(K_a)^{-1} \int_{(R_{\bar{Q}}(F)\backslash R(F))^2} f(h,h')dhdh'$$

where

$$f(h,h') = \int_{K_a} \mathcal{L}_\tau(e_\varphi(h), e_{\varphi'}(h'))\omega(h^{-1}h')\psi(t\lambda(h'))\psi(-t\lambda(h)) \mid t \mid^{-1} dt.$$

Combining with the definition of e_φ and $e_{\varphi'}$, we have

$$\begin{aligned} \mathcal{L}_\pi(e_\varphi, e_{\varphi'}) &= meas(K_a)^{-1}\mathcal{L}_\tau(v_0,v_0') \\ &\times \int_{\mathcal{U}^2}\int_{K_a} f(h)\overline{f'(h')}\omega(s(h)^{-1}s(h'))\psi(t\lambda(s(h)))\psi(-t\lambda(s(h))) \mid t \mid^{-1} dtdhdh'. \end{aligned}$$

Now if we take $\mathcal{U}$ small enough, we can choose a suitable section $s : \mathcal{U} \to R(F)$ such that for all $t \in K_a$ and $h \in s(\mathcal{U})$, we have $\psi(t\lambda(h)) = \omega(h) = 1$. Also by taking K_0 small, we may assume that $\mid t \mid = 1$ for all $t \in K_a$. Then the integral above becomes

$$\begin{aligned} \mathcal{L}_\pi(e_\varphi, e_{\varphi'}) &= meas(K_a)^{-1}\mathcal{L}_\tau(v_0,v_0') \int_{\mathcal{U}^2}\int_{K_a} f(h)\overline{f'(h')}dtdhdh' \\ &= \mathcal{L}_\tau(v_0,v_0')\int_{\mathcal{U}^2} f(h)\overline{f'(h')}dhdh'. \end{aligned}$$

Thus we can easily choose f and f' so that $\mathcal{L}_\pi(e_\varphi, e_{\varphi'}) \neq 0$. Therefore we have proved that $\mathcal{L}_\tau \neq 0 \Rightarrow \mathcal{L}_\pi \neq 0$. □

6.4 Parabolic Induction for the archimedean Case

Assume that F is archimedean in this section. It is very hard to directly study any arbitrary parabolic induction because of the way that we normalize the integral. Instead, we first study the parabolic induction for $\bar{P}$, then study all other parabolic subgroups contained in $\bar{P}$. This is allowable since in the archimedean case, the discrete series only appear on $\mathrm{GL}_1(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{R})$, $\mathrm{GL}_1(D)$ and $\mathrm{GL}_1(\mathbb{C})$. Let π be a tempered representation of G with central character η. Since we are in archimedean case, there exists a tempered representation π_0 of G_0 such that $\pi = I_{\bar{P}}^G(\pi_0)$. We assume that the inner product on π is given by

$$(e, e') = \int_{\bar{P}(F)\backslash G(F)} (e(g), e'(g))_{\pi_0} dg, \ e, e' \in \pi = I_{\bar{P}}^G(\pi_0). \tag{6.13}$$

For $T \in \mathrm{End}(\pi_0)^\infty$, define

$$\mathcal{L}_{\pi_0}(T) = \int_{Z_H(F)\backslash H(F)} \mathrm{tr}(\pi_0(h_0^{-1})T)\omega(h_0)dh_0.$$

The integral above is absolutely convergent by Lemma 4.3.1(1) together with the fact that π_0 is tempered.

Proposition 6.4.1. *With the notation above, we have*

$$\mathcal{L}_\pi \neq 0 \iff \mathcal{L}_{\pi_0} \neq 0.$$

Proof. **We first consider the case when $F = \mathbb{R}$.** For $e, e' \in \pi^\infty$, we have

$$\mathcal{L}_\pi(e, e') = \int_{Z_R(F)\backslash R(F)}^* \int_{\bar{P}(F)\backslash G(F)} (e(g), e'(gh))dg\xi(h)\omega(h)dh.$$

Same as in Proposition 6.1.1, we can find $\varphi_1 \in C_c^{2m-2}(F^\times)$ and $\varphi_2 \in C_c^\infty(F^\times)$ such that $\varphi_1 * \Delta^m + \varphi_2 = \delta_1$, and we have

$$\begin{aligned} \mathcal{L}_\pi(e, e') \ = \ & \int_{Z_R(F)\backslash R(F)} Ad_a(\Delta^m)(\int_{\bar{P}(F)\backslash G(F)} (e(g), e'(gh))dg) \\ & \times \int_F \varphi_1(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdh \\ & + \int_{Z_R(F)\backslash R(F)} \int_{\bar{P}(F)\backslash G(F)} (e(g), e'(gh)) \\ & \times \int_F \varphi_2(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdgdh. \end{aligned} \tag{6.14}$$

Here $Ad_a(\Delta^m)$ acts on the function $\int_{\bar{P}(F)\backslash G(F)}(e(g), e'(gh))dg$ for the variable h. It is clear that this action commutes with the integral $\int_{\bar{P}(F)\backslash G(F)}$. Also since $\bar{P}$ is a good parabolic subgroup, by Proposition 4.2.1, we can choose Haar measure compatibly so that for all $\varphi \in L_1(\bar{P}(F)\backslash G(F), \delta_{\bar{P}})$, we have

$$\int_{\bar{P}(F)\backslash G(F)} \varphi(g)dg = \int_{U(F)} \varphi(h)dh.$$

Therefore (6.14) becomes

$$\begin{aligned}
\mathcal{L}_\pi(e,e') &= \int_{Z_R(F)\backslash R(F)} \int_{U(F)} Ad_a(\Delta^m)((e(u),e'(uh)))du \\
&\quad \times \int_F \varphi_1(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdh \\
&\quad + \int_{Z_R(F)\backslash R(F)} \int_{U(F)} (e(u),e'(uh)) \\
&\quad \times \int_F \varphi_2(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdudh.
\end{aligned}$$

Here $Ad_a(\Delta^m)$ acts on the function $(e(u),e'(uh))$ for the variable h. By changing the order of integration $\int_{Z_R(F)\backslash R(F)} \int_{U(F)}$ and decomposing the integral $\int_{Z_R(F)\backslash R(F)}$ by $\int_{U(F)} \int_{Z_H(F)\backslash H(F)}$ (this is allowable since the outer two integrals are absolutely convergent by Proposition 6.1.1), together with the fact that Ad_a is the identity map on H, we have

$$\begin{aligned}
\mathcal{L}_\pi(e,e') &= \int_{U(F)} \int_{U(F)} Ad_a(\Delta^m)(\mathcal{L}_{\pi_0}(e(u),e'(uu')))\varphi_1'(\lambda(u'))du'du \\
&\quad + \int_{U(F)} \int_{U(F)} \mathcal{L}_{\pi_0}(e(u),e'(uu'))\varphi_2'(\lambda(u'))du'du
\end{aligned}$$

where $\varphi_i'(s) = \int_F \varphi_i(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(ts)dt$ is the Fourier transforms of the function $\varphi_i(t)\delta_P(a(t)) \mid t \mid^{-1}$ for $i=1,2$. Here $Ad_a(\Delta^m)$ acts on the function $\mathcal{L}_{\pi_0}(e(u),e'(uu'))$ for the variable u'. In particular, this implies $\mathcal{L}_\pi \neq 0 \Rightarrow \mathcal{L}_{\pi_0} \neq 0$.

For the other direction, if $\mathcal{L}_{\pi_0} \neq 0$, we can choose $v_1, v_2 \in \pi_0^\infty$ such that $\mathcal{L}_{\pi_0}(v_1,v_2) \neq 0$. Choose $f_1, f_2 \in C_c^\infty(U(F))$, for $i=1,2$, similarly as in the p-adic case, define

$$e_{f_i}(g) = \begin{cases} \delta_{\bar{P}}(l)\pi_0(l)f_i(u)v_i & \text{if} \quad g = l\bar{u}u \textit{ with } l \in G_0(F), u \in U(F), \bar{u} \in \bar{U}(F); \\ 0 & \text{else.} \end{cases}$$

These are elements in π^∞, and we have

$$\begin{aligned}
\mathcal{L}_\pi(e_{f_1},e_{f_2}) &= \int_{U(F)} \int_{U(F)} \mathcal{L}_{\pi_0}(v_1,v_2)f_1(u)Ad_a(\Delta^m)(f_2(uu')))\varphi_1'(\lambda(u'))du'du \\
&\quad + \int_{U(F)} \int_{U(F)} \mathcal{L}_{\pi_0}(v_1,v_2)f_1(u)f_2(uu')\varphi_2'(\lambda(u'))du'du. \qquad (6.15)
\end{aligned}$$

Here $Ad_a(\Delta^m)$ acts on the function $f_2(uu')$ for the variable u'. Then we can easily find f_1, f_2 such that (6.15) is non-zero. This proves that $\mathcal{L}_{\pi_0} \neq 0 \Rightarrow \mathcal{L}_\pi \neq 0$, and finishes the proof of the proposition for the case when $F = \mathbb{R}$.

If $F = \mathbb{C}$, the argument is similar to the real case and we will skip it here. $\square$

Now for a tempered representation π_0 of $G_0(F)$ whose central character equals η when restricting on Z_G, we can find a good parabolic subgroup $\bar{Q}_0 = L_0U_0$ of $G_0(F)$ and a discrete series τ of L_0 such that $\pi_0 = I_{\bar{Q}_0}^{G_0}(\tau)$. We still assume that the inner product on π_0 is given by

$$(e, e') = \int_{\bar{Q}_0(F)\backslash H_0(F)} (e(g), e'(g))_\tau dg,\ e, e' \in \pi_0 = I_{\bar{Q}_0}^{G_0}(\tau). \tag{6.16}$$

Let $H_{\bar{Q}} = H \cap \bar{Q}_0$. For $T \in \mathrm{End}(\tau)^\infty$, define

$$\mathcal{L}_\tau(T_\tau) = \int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \mathrm{tr}(\tau(h_{\bar{Q}}^{-1})T)\delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}\omega(h_{\bar{Q}})dh_{\bar{Q}}.$$

The integral above is absolutely convergent by Proposition 4.4.2(2) together with the assumption that τ is discrete series.

Proposition 6.4.2. *With the notation above, we have*

$$\mathcal{L}_{\pi_0} \neq 0 \iff \mathcal{L}_\tau \neq 0.$$

Proof. Since we are in (G_0, H) case, the integral defining $\mathcal{L}_{\pi_0}$ is absolutely convergent. Together with (6.16), we have

$$\mathcal{L}_{\pi_0}(e, e') = \int_{Z_H(F)\backslash H(F)} \int_{\bar{Q}_0(F)\backslash G_0(F)} (e(g), e'(gh))_\tau \omega(h) dg dh.$$

The integral above is absolutely convergent by Lemma 4.3.1. Same as in the previous Propositions, the integral $\bar{Q}_0(F)\backslash G_0(F)$ can be replaced by $H_{\bar{Q}}(F)\backslash H(F)$, hence we have

$$\mathcal{L}_{\pi_0}(e, e') = \int_{Z_H(F)\backslash H(F)} \int_{H_{\bar{Q}}(F)\backslash H(F)} (e(h'), e'(h'h))_\tau \omega(h) dh' dh.$$

By switching the two integrals, changing the variable $h \to h'h$ and decomposing the integral $\int_{Z_H(F)\backslash H(F)}$ by $\int_{H_{\bar{Q}}(F)\backslash H(F)} \int_{Z_H(F)\backslash H_{\bar{Q}}(F)}$, we have

$$\mathcal{L}_\pi(e, e') = \int_{H_{\bar{Q}}(F)\backslash H(F)} \int_{H_{\bar{Q}}(F)\backslash H(F)} \mathcal{L}_\tau(e(h), e'(h'))\omega(h)^{-1}\omega(h') dh dh'.$$

This proves $\mathcal{L}_{\pi_0}\neq 0\Rightarrow \mathcal{L}_{\tau}\neq 0$.

For the other direction, if $\mathcal{L}_{\tau}\neq 0$, there exist $v_1,v_2\in\tau^{\infty}$ such that $\mathcal{L}_{\tau}(v_1,v_2)\neq 0$. Let $s:\mathcal{U}\rightarrow H(F)$ be an analytic section over an open subset $\mathcal{U}$ of $H_{\bar{Q}}(F)\backslash H(F)$ of the map $H(F)\rightarrow H_{\bar{Q}}(F)\backslash H(F)$. Choose $f_1,f_2\in C_c^{\infty}(\mathcal{U})$, for $i=1,2$, define

$$e_{f_i}(g)=\begin{cases}\delta_{\bar{Q}}(l)\tau(l)f_i(h)v_i & \text{if} \quad g=lus(h)\ with\ l\in L_0(F),u\in U_0(F),h\in\mathcal{U};\\ 0 & \text{else.}\end{cases}$$

These are elements in π_0^{∞}, and we have

$$\mathcal{L}_{\pi_0}(e_{f_1},e_{f_2})=\int_{\mathcal{U}}\int_{\mathcal{U}}f_1(h)\overline{f_2}(h')\omega(s(h))^{-1}\omega(s(h'))\mathcal{L}_{\tau}(v_1,v_2)dhdh'.$$

Then we can easily choose f_1,f_2 such that $\mathcal{L}_{\pi_0}(e_{f_1},e_{f_2})\neq 0$. This proves the other direction, and finishes the proof of the Proposition. $\square$

Now let π be a tempered representation of $G(F)$. Then we can find a good parabolic subgroup $L_0U_0=\bar{Q}\subset\bar{P}(F)$ and a discrete series τ of L_0, such that $\pi=I_{\bar{Q}}^{G}(\tau)$ (note that we are in archimedean case, only $GL_1(F),GL_2(F)$ and $GL_1(D)$ have discrete series). Combining Proposition 6.4.1 and Proposition 6.4.2, we have the following Proposition.

Proposition 6.4.3. *With the notation above, we have*

$$\mathcal{L}_{\pi}\neq 0\iff\mathcal{L}_{\tau}\neq 0.$$

6.5 Proof of Theorem 6.2.1

Let π be a tempered representation of $G(F)$ with central character η. We already know $\mathcal{L}_{\pi}\neq 0\Rightarrow m(\pi)\neq 0$. We are going to prove the other direction. If $F=\mathbb{C}$, π is always a principal series. In other words, we can find an unitary character τ of the torus such that π is the parabolic induction of τ. It is easy to see from the definition that $\mathcal{L}_{\tau}(T)=\mathrm{tr}(T)$ for $T\in\mathrm{End}(\tau)^{\infty}$. Therefore $\mathcal{L}_{\tau}\neq 0$, which implies $\mathcal{L}_{\pi}\neq 0$ by Proposition 6.4.3. This tells us that $m(\pi)$ and $\mathcal{L}_{\pi}$ are always nonzero if $F=\mathbb{C}$. This proves Theorem 6.2.1.

If $F\neq\mathbb{C}$ and $m(\pi)\neq 0$, let $0\neq l\in Hom_H(\pi^{\infty},\xi)$. We first prove

(1) For all $e \in \pi^\infty$ and $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, the integral

$$\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg \tag{6.17}$$

is absolutely convergent.

In fact, this is equivalent to the convergence of

$$\int_{R(F)\backslash G(F)} \mid l(\pi(x)e) \mid \int_{Z_R(F)\backslash R(F)} \mid f(hx) \mid dhdx.$$

By Proposition 4.4.1, for all $d > 0$ and $x \in R(F)\backslash G(F)$, we have

$$\int_{Z_R(F)\backslash R(F)} \mid f(hx) \mid dh \ll \Xi^{R\backslash G}(x)\sigma_{R\backslash G}(x)^{-d}. \tag{6.18}$$

On the other hand, by Lemma 6.2.3, there exists $d' > 0$ such that for all $x \in R(F)\backslash G(F)$, we have

$$\mid l(\pi(x)e) \mid \ll \Xi^{R\backslash G}(x)\sigma_{R\backslash G}(x)^{d'}. \tag{6.19}$$

Then (1) follows from (6.18) and (6.19), together with Proposition 4.4.1.

Now we can compute (6.17) in two different ways. First, since $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1}) = C_c^\infty(Z_G(F)\backslash G(F), \eta^{-1}) * \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, we can write $f = \varphi * f'$ for some $\varphi \in C_c^\infty(Z_G(F)\backslash G(F), \eta^{-1})$ and $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$. Then

$$\begin{aligned}
&\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg \\
= &\int_{Z_G(F)\backslash G(F)} \int_{Z_G(F)\backslash G(F)} l(\pi(g)e)\varphi(g')f'(g'^{-1}g)dg'dg \\
= &\int_{Z_G(F)\backslash G(F)} \int_{Z_G(F)\backslash G(F)} l(\pi(g'g)e)\varphi(g')dg'f'(g)dg \\
= &\int_{Z_G(F)\backslash G(F)} l(\pi(\varphi)\pi(g)e)f'(g)dg.
\end{aligned}$$

Since the vector $l \circ \pi(\varphi) \in \pi^{-\infty}$ belongs to $\bar{\pi}^\infty$, by the definition of the action of $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ on π^∞, we have

$$\begin{aligned}
&\int_{Z_G(F)\backslash G(F)} l(\pi(\varphi)\pi(g)e)f'(g)dg \\
= &\int_{Z_G(F)\backslash G(F)} f'(g)(\pi(g)e, l\cdot\pi(\varphi))dg \\
= &(\pi(f')e, l\cdot\pi(\varphi)) = l(\pi(\varphi)\pi(f')e) = l(\pi(f)e).
\end{aligned}$$

This tells us

$$\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg = l(\pi(f)e). \tag{6.20}$$

On the other hand,

$$\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg = \int_{R(F)\backslash G(F)} l(\pi(x)e) \int_{Z_R(F)\backslash R(F)} f(hx)\xi(h)\omega(h)dhdx.$$

By Lemma 6.2.2(4), if the map $\Pi \in \Pi_{temp}(G,\eta) \to \Pi(f)$ is compactly supported, we have

$$\begin{aligned} & \int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg \\ = & \int_{R(F)\backslash G(F)} l(\pi(x)e) \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\Pi(\Pi(f)\Pi(x^{-1}))\mu(\Pi)d\Pi dx. \end{aligned} \tag{6.21}$$

For $T \in C_c^\infty(\Pi_{temp}(G,\eta))$, by applying (6.20) and (6.21) to the function $f = f_T$, we have

$$l(T_\pi e) = \int_{R(F)\backslash G(F)} l(\pi(x)e) \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\Pi(T_\Pi \Pi(x^{-1}))\mu(\Pi)d\Pi dx \tag{6.22}$$

for all $e \in \pi^\infty$. Now assume that $\pi = I_Q^G(\sigma)$ for some good parabolic subgroup $Q = LU_Q$ of G and some $\sigma \in \Pi_2(L)$. Let

$$\mathcal{O} = \{Ind_Q^G(\sigma_\lambda) \mid \lambda \in i\mathfrak{a}_{L,0}^*\} \subset \Pi_{temp}(G,\eta)$$

be the connected component containing π. Choose $e_0 \in \pi^\infty$ such that $l(e_0) \neq 0$, and let $T_0 \in \mathrm{End}(\pi)^\infty$ with $T_0(e_0) = e_0$. We can easily find an element $T^0 \in C_c^\infty(\Pi_{temp}(G,\eta))$ such that

$$T_\pi^0 = T_0,\ Supp(T^0) \subset \mathcal{O}.$$

By applying (6.22) to the case that $e = e_0, T = T^0$, we know there exists $\lambda \in i\mathfrak{a}_{L,0}^*$ such that $\mathcal{L}_{\pi_\lambda} \neq 0$ where $\pi_\lambda = Ind_Q^G(\sigma_\lambda)$. By Proposition 6.3.1 and Proposition 6.4.3, this implies $\mathcal{L}_{\sigma_\lambda} \neq 0$. We need a Lemma:

Lemma 6.5.1. *For all $\lambda \in i\mathfrak{a}_{L,0}^*$, we have*

$$\mathcal{L}_\sigma \neq 0 \iff \mathcal{L}_{\sigma_\lambda} \neq 0.$$

Proof. **We first assume that F is p-adic.** If π itself is a discrete series, $\sigma = \pi$ and $G = Q$. Then the lemma just follows from the definition of $\mathcal{L}_\pi$. If $Q \neq G$, we are in the reduced models case. If the reduced model is of Type I, there are two models: the middle model and the trilinear GL_2 model. For those models, it is easy to show (just by the definition) that the nonvanishing property of $\mathcal{L}_\sigma$ is invariant under the unramified twist.

For type II models, it is not clear from the definition that the unramified twist will preserve the nonvanishing property. However, we can prove it by proving a much strongly argument. We claim that for all Type II reduced models, $\mathcal{L}_\sigma$ is always nonzero for all discrete series σ. In fact, by applying the same argument above to the reduced model, we can have a similar formula as (6.22) for $\mathcal{L}_\sigma$. Since σ is a discrete series, the connected component containing it does not contains other element (i.e. $\mathcal{O} = \{\sigma\}$). Then by applying the same argument above, we know that $m(\sigma) \neq 0 \Rightarrow \mathcal{L}_\sigma \neq 0$ (**The upshot is that since σ is a discrete series, we don't need to worry about the unramified twist issue**). Therefore we only need to show that for all type II models, the multiplicity $m(\sigma)$ is always nonzero. This has already been proved in Theorem 5.4.2. This finishes the proof of the lemma.

If $F = \mathbb{R}$, we will prove the lemma in Section 7.3. □

Now by applying Lemma 6.5.1, we know $\mathcal{L}_\sigma \neq 0$. Applying Proposition 6.3.1 and Proposition 6.4.3 again, we have $\mathcal{L}_\pi \neq 0$. This proves the other direction, and finishes the proof of Theorem 6.2.1.

6.6 Some Consequences

If $F = \mathbb{C}$, the following Corollary has already been proved in the previous section.

Corollary 6.6.1. *For all tempered representations π of $G(F)$ with central character η, we have*

$$\mathcal{L}_\pi \neq 0, \ m(\pi) \neq 0.$$

In particular, since $m(\pi) \leq 1$, we have

$$m(\pi) = 1.$$

If $F = \mathbb{R}$, let π be a tempered representation of $G(F)$ with central character η. Since we are in the archimedean case, there exists a tempered representation π_0 of G_0 such that $\pi = I_{\bar{P}}^{G}(\pi_0)$. We have the following result.

Corollary 6.6.2. $m(\pi) = m(\pi_0)$.

Proof. Similar to Theorem 6.2.1, we have

$$m(\pi_0) \neq 0 \iff \mathcal{L}_{\pi_0} \neq 0.$$

Then by applying Proposition 6.4.1, we have

$$m(\pi) \neq 0 \iff \mathcal{L}_{\pi} \neq 0 \iff \mathcal{L}_{\pi_0} \neq 0 \iff m(\pi_0) \neq 0.$$

By the strong multiplicity one theorem, $m(\pi)$ and $m(\pi_0)$ are either 1 or 0. Then the above equivalence just tells us $m(\pi) = m(\pi_0)$. □

If F is p-adic, let π be a tempered representation of $\mathrm{GL}_6(F)$ with central character η. We can find a good parabolic subgroup $\bar{Q} = LU_Q$ and a discrete series σ of $L(F)$ such that $\pi = I_{\bar{Q}}^{G}(\sigma)$. By the construction of the local Jacquet-Langlands correspondence, we know that $\pi_D \neq 0$ iff $\bar{Q}$ is of Type I or $\bar{Q} = G$. In fact, the local Jacquet-Langlands correspondence established in [DKV84] gives a bijection between the discrete series series. Then the map can be extended naively to all the tempered representations via the parabolic induction since all tempered representations are full induced from some discrete series (note that we are in the GL_n case). Therefore, in order to make $\pi_D \neq 0$, the Levi subgroup L should have an analogy in $\mathrm{GL}_3(D)$, which is equivalent to say that $\bar{Q}$ is of Type I or $\bar{Q} = G$.

Corollary 6.6.3. *If $\bar{Q}$ is of type II, Theorem 1.2.1 holds.*

Proof. By the discussion above, we know $\pi_D = 0$, so we only need to show that $m(\pi) = 1$. By the strong multiplicity one theorem, we only need to show that $m(\pi) \neq 0$. By the proof of Lemma 6.5.1, we know $\mathcal{L}_\sigma \neq 0$. Together with Proposition 6.3.1, we have $\mathcal{L}_\pi \neq 0$. By Theorem 6.2.1, this implies $m(\pi) \neq 0$ and this proves the Corollary. □

Now let π be a tempered representation of $G(F)$ with central character η (note that $G(F)$ can be both $\mathrm{GL}_6(F)$ and $\mathrm{GL}_3(D)$), we can find a good parabolic subgroup $\bar{Q} = LU_Q$ and a discrete series σ of $L(F)$ such that $\pi = I_{\bar{Q}}^{G}(\sigma)$. **We assume that $\bar{Q}$ is of Type I or $\bar{Q} = G$.**

Corollary 6.6.4. *1.* $m(\pi) = m(\sigma)$.

2. Let $\mathcal{K} \subset \Pi_{temp}(G, \eta)$ *be a compact subset. Then there exists an element* $T \in \mathcal{C}(\Pi_{temp}(G, \eta))$ *such that* $\mathcal{L}_\pi(T_\pi) = m(\pi)$ *for all* $\pi \in \mathcal{K}$.

Proof. (1) follows from the same proof as in Corollary 6.6.2. For (2), it is enough to show that for all $\pi' \in \Pi_{temp}(G, \eta)$, there exists $T \in \mathcal{C}(\Pi_{temp}(G, \eta))$ such that $\mathcal{L}_\pi(T_\pi) = m(\pi)$ for all π in some neighborhood of π' in $\Pi_{temp}(G, \eta)$. Since $m(\sigma)$ is invariant under the unramified twist for type I models, combining with part (1) and Corollary 6.6.3, we know that the map $\pi \to m(\pi)$ is locally constant (In fact, we even know that the map is constant on each connected components of $\Pi_{temp}(G, \eta)$). If $m(\pi') = 0$, we can just take $T = 0$, and there is nothing to prove.

If $m(\pi') \neq 0$, then we know $m(\pi) = 1$ for all π in the connected component containing π'. By Theorem 6.2.1, we can find $T' \in \mathrm{End}(\pi')^\infty$ such that $\mathcal{L}_{\pi'}(T') \neq 0$. Then let $T^0 \in \mathcal{C}(\Pi_{temp}(G, \eta))$ be an element with $T^0_{\pi'} = T'$. By Lemma 6.2.2(1), the function $\pi \to \mathcal{L}_\pi(T^0_\pi)$ is a smooth function. The value at π' is just $\mathcal{L}_{\pi'}(T') \neq 0$. As a result, we can find a smooth and compactly supported function φ on $\Pi_{temp}(G, \eta)$ such that $\varphi(\pi)\mathcal{L}_\pi(T^0_\pi) = 1$ for all π belonging to a small neighborhood of π'. Then we just need to take $T = \varphi T^0$ and this proves the Corollary. $\square$

Chapter 7

The Archimedean Case

In this chapter, we will prove our main theorems (i.e. Theorem 1.2.1 and Theorem 1.2.2) when the field F is archimedean. In Section 7.1, we will prove the complex case. In Section 7.2, we will give a brief review of the trilinear GL_2 models. Then in Section 7.3, we will prove the real case. The main ingredient of the proof is Corollary 6.6.2, which allows us to reduce the problem to the trilinear GL_2 model case. Then by applying the results of Prasad ([P90]) and Loke ([L01]), we can prove the two main theorems.

7.1 The Complex Case

In this section, we assume that $F = \mathbb{C}$. In this case, π_D is always 0. As a result, in order to prove Theorem 1.2.1 and Theorem 1.2.2, we only need to prove the following proposition.

Proposition 7.1.1. *Let π be an irreducible tempered representation of $G(F)$ with central character χ^2. The followings hold.*

1. $m(\pi) = 1$.
2. *If the central character of π is trivial, then we have*

$$\epsilon(1/2, \pi, \wedge^3) = 1.$$

Proof. (1) has already been proved in Corollary 6.6.1. For (2), since we are in the complex case, every tempered representation is a principal series. Hence we can find a tempered representation $\sigma = \sigma_1 \otimes \sigma_2$ of $GL_5(F) \times \mathrm{GL}_1(F)$ such that π is the parabolic induction of σ. Let ϕ be the Langlands parameter of π, and let ϕ_i be the Langlands parameter of σ_i for $i = 1, 2$. Then we have $\phi = \phi_1 \oplus \phi_2$, and this implies

$$\wedge^3(\phi) \quad = \quad \wedge^3(\phi_1 \oplus \phi_2) = \wedge^3(\phi_1) \oplus (\wedge^2(\phi_1) \otimes \phi_2).$$

Since the central character of π is trivial, $\det(\phi) = \det(\phi_1) \otimes \det(\phi_2) = 1$. Therefore $(\wedge^3(\phi_1))^\vee = \wedge^2(\phi_1) \otimes \det(\phi_1)^{-1} = \wedge^2(\phi_1) \otimes \det(\phi_2) = \wedge^2(\phi_1) \otimes \phi_2$. Hence

$$\epsilon(1/2, \pi, \wedge^3) = \det(\wedge^3(\phi_1))(-1) = (\det(\phi_1))^6(-1) = 1.$$

This finishes the proof of the proposition. $\square$

7.2 The Trilinear GL_2 Models

In this subsection, we recall Prasad's result on the trilinear GL_2 model. For the rest two sections of this chapter, **we assume that** $F = \mathbb{R}$**.** Let $G_0 = GL_2(F) \times GL_2(F) \times GL_2(F)$, $H = GL_2(F)$ diagonally embed into G_0. For a given irreducible representation $\pi_0 = \pi_1 \otimes \pi_2 \otimes \pi_3$ of G_0, assume that the central character of π_0 equals χ^2 on $Z_H(F)$ for some unitary character χ of $F^\times$. χ will induce an one-dimensional representation ω_0 of H. Let

$$m(\pi_0) = dim(Hom_{H(F)}(\pi_0, \omega_0)). \tag{7.1}$$

Similarly, we have the quaternion algebra version: let $G_{0,D} = GL_1(D) \times GL_1(D) \times GL_1(D)$, and let $H_D = GL_1(D)$. We can still define the multiplicity $m(\pi_{0,D})$. The following theorem has been proved by Prasad in his thesis [P90] under the assumption that at least one π_i is discrete series (i=1,2,3), and by Loke in [L01] for the case when π_0 is a principal series.

Theorem 7.2.1. *With the notation above, if π_0 is an irreducible generic representation of G_0, let $\pi_{0,D}$ be the Jacquet-Langlands correspondence of π_0 to $G_{0,D}$ if it exists; otherwise let $\pi_{0,D} = 0$. Then we have*

1. *$m(\pi_0) + m(\pi_{0,D}) = 1$.*
2. *If the central character of π is trivial on $Z_H(F)$, then*

$$m(\pi_0) = 1 \iff \epsilon(1/2, \pi_0) = 1$$

and

$$m(\pi_{0,D}) = 1 \iff \epsilon(1/2, \pi_0) = -1.$$

Remark 7.2.2. *Both Prasad's result and Loke's result are based on the assumption that the product of the central characters of π_i ($i = 1, 2, 3$) is trivial. In our case, we assume that the product of the central characters is χ^2. But we can always reduce our case to their cases by replacing π_1 with $\pi_1 \otimes (\chi^{-1} \circ \det)$. Note that twist by characters will not change the multiplicity. On the other hand, for the epsilon factor part, we do need the assumption that the product of the central character is trivial. Otherwise the Langlands parameter of π_0 will no longer be selfdual, hence the value of the epsilon factor at $1/2$ may not be ± 1.*

7.3 The Real Case

Let π be an irreducible tempered representation of $GL_6(F)$, with $F = \mathbb{R}$. There exists a tempered representation π_0 of G_0 such that $\pi = Ind_{\bar{P}}^{G}(\pi_0)$. Let π_D be the Jacquet-Langlands correspondence of π to $GL_3(D)$. Similarly we can find a tempered representation $\pi_{0,D}$ of $G_{0,D}$ such that $\pi = Ind_{\bar{P}_D}^{G_D}(\pi_{0,D})$. It is easy to see that $\pi_{0,D}$ is the Jacquet-Langlands correspondence of π_0 to $G_{0,D}$. Note that π_D and $\pi_{0,D}$ may be zero. In fact, they are nonzero if and only if π_0 is a discrete series. By Corollary 6.6.2, $m(\pi) = m(\pi_0)$ and $m(\pi_D) = m(\pi_{0,D})$. Then by applying Theorem 7.2.1, we have $m(\pi) + m(\pi_D) = m(\pi_0) + m(\pi_{0,D}) = 1$. This proves Theorem 1.2.1.

For Theorem 1.2.2, by Theorem 7.2.1, it is enough to show that

$$\epsilon(1/2, \pi, \wedge^3) = \epsilon(1/2, \pi_0).$$

For $i = 1, 2, 3$, let ϕ_i be the Langlands parameter of π_i. Then the Langlands parameter of π is $\phi_{\pi_0} = \phi_1 \oplus \phi_2 \oplus \phi_3$. This implies

$$\begin{aligned} & \wedge^3(\phi_{\pi_0}) = \wedge^3(\phi_1 \oplus \phi_2 \oplus \phi_3) \\ = \quad & (\phi_1 \otimes \phi_2 \otimes \phi_3) \oplus (\det(\phi_2) \otimes \phi_1) \oplus (\det(\phi_3) \otimes \phi_1) \\ & \oplus (\det(\phi_1) \otimes \phi_2) \oplus (\det(\phi_3) \otimes \phi_2) \oplus (\det(\phi_1) \otimes \phi_3) \oplus (\det(\phi_2) \otimes \phi_3). \end{aligned}$$

By our assumption on the central character, we have $\det(\phi_{\pi_0}) = \det(\phi_1) \otimes \det(\phi_2) \otimes \det(\phi_3) = 1$. Therefore $(\det(\phi_2) \otimes \phi_1)^\vee = \det(\phi_1)^{-1} \otimes \det(\phi_2)^{-1} \otimes \phi_1 = \det(\phi_3) \otimes \phi_1$.

This implies

$$\epsilon(1/2, \det(\phi_2)\otimes\phi_1)\epsilon(1/2, \det(\phi_3)\otimes\phi_1) = \det(\phi_1)\otimes\det(\phi_2)^2(-1) = \det(\phi_1)(-1).$$

Similarly, we have

$$\begin{aligned}\epsilon(1/2, \det(\phi_1)\otimes\phi_2)\epsilon(1/2, \det(\phi_3)\otimes\phi_2) &= \det(\phi_1)^2\otimes\det(\phi_2)(-1) = \det(\phi_2)(-1),\\ \epsilon(1/2, \det(\phi_1)\otimes\phi_3)\epsilon(1/2, \det(\phi_2)\otimes\phi_3) &= \det(\phi_1)^2\otimes\det(\phi_3)(-1) = \det(\phi_3)(-1).\end{aligned}$$

Combining the three equations above, we have

$$\begin{aligned}\epsilon(1/2, \pi, \wedge^3) &= \det(\phi_1)\otimes\det(\phi_2)\otimes\det(\phi_3)(-1)\epsilon(1/2, \phi_1\otimes\phi_2\otimes\phi_3)\\ &= \epsilon(1/2, \phi_1\otimes\phi_2\otimes\phi_3) = \epsilon(1/2, \pi_0).\end{aligned}$$

This proves Theorem 1.2.2.

Now the only thing left is to prove Lemma 6.5.1 for the case when $F = \mathbb{R}$. As in the p-adic case, for Type I models, the lemma just follows from the definition of $\mathcal{L}_\sigma$. For Type II models, as in the p-adic case, we only need to prove that the multiplicity is always nonzero. Since $F = \mathbb{R}$, only $\mathrm{GL}_2(F)$ and $\mathrm{GL}_1(F)$ have discrete series. As a result, there are only three Type II models: Type $(2,2,1,1)$, $(2,1,1,1,1)$ and $(1,1,1,1,1,1)$. Type $(1,1,1,1,1,1)$ case is trivial since L and $H_{\bar{Q}}$ are both abelian groups in this case. For Type $(2,1,1,1,1)$, by canceling the GL_1 part (which is abelian), we are considering the model $(\mathrm{GL}_2(F), T)$ where $T = \{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} | a, b \in F^\times\}$ is the maximal torus. This is the Bessel model for $(\mathrm{GL}_2, \mathrm{GL}_1)$, and we know the multiplicity is always nonzero by the archimedean Rankin-Selberg theory of Jacquet and Shalika ([JS90]).

For Type $(2,2,1,1)$, by canceling the GL_1 part, we are considering the following model: $M = \mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$, and

$$M_0 = \{m(a,b) = \begin{pmatrix} a & 0\\ c & b\end{pmatrix}\times\begin{pmatrix} a & 0\\ c & b\end{pmatrix} | a, b\in F^\times,\ c\in F\}.$$

The character on M_0 is given by $\omega(m(a,b)) = \chi(ab)$. Let B be the lower Borel subgroup of $\mathrm{GL}_2(F)$. It is isomorphic to M_0, hence we can also view ω as a character on B. Let $\pi_3 = I_B^G(\omega)$, it is a principal series of $\mathrm{GL}_2(F)$. For any irreducible tempered representation $\pi_1\otimes\pi_2$ of M, by the Frobenius reciprocity, we have

$$Hom_{M_0}(\pi_1\otimes\pi_2, \omega) = Hom_{\mathrm{GL}_2(F)}(\pi_1\otimes\pi_2, \pi_3).$$

Here $\mathrm{GL}_2(F)$ maps diagonally into M. Therefore the Hom space is isomorphic to the Hom space of the trilinear GL_2 model for the representation $\pi_0 = \pi_1 \otimes \pi_2 \otimes \pi_3$. Since π_3 is a principal series, $\pi_{0,D} = 0$. By Theorem 7.2.1, $m(\pi_0) = 1 \neq 0$, hence the Hom space is nonzero and this proves Lemma 6.5.1. **Now the proof of our main theorems (Theorem 1.2.1 and Theorem 1.2.2) is complete for the archimedean case.**

Chapter 8

The Proof of the Spectral Side of the Trace Formula

For the rest of this paper, we assume that F is p-adic except for Chapter 14 and 15. In this chapter, we will prove the spectral side of the trace formula. In Section 8.1, we will prove the integral defining $I(f)$ is absolutely convergent. We will postpone the proof of a technical proposition (i.e Proposition 8.1.1) to Appendix B. Then in Section 8.2, we prove the spectral expansion.

8.1 Absolutely Convergence of $I(f)$

Let $\eta = \chi^2$ be two unitary characters of $F^{\times}$. For $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, as in Chapter 5, define the function $I(f, \cdot)$ on $R(F)\backslash G(F)$ to be

$$I(f,x) = \int_{Z_R(F)\backslash R(F)} f(x^{-1}hx)\xi(h)\omega(h)dh.$$

By Lemma 4.3.1(2), the above integral is absolutely convergent. The following Proposition together with Proposition 4.4.1(3) tell us that the integral

$$I(f) := \int_{R(F)\backslash G(F)} I(f,x)dx$$

is also absolutely convergent for all $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$, and this defines a continuous linear form

$$\mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1}) \to \mathbb{C} : \ f \to I(f).$$

In particular, this implies

$$\lim_{N\to\infty} I_N(f) = I(f). \tag{8.1}$$

Proposition 8.1.1. *1. There exist $d > 0$ and a continuous semi-norm ν on $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ such that*

$$|I(f,x)| \leq \nu(f)\Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^d$$

for all $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ and $x \in R(F)\backslash G(F)$.

2. For all $d > 0$, there exists a continuous semi-norm ν_d on $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ such that

$$|I(f,x)| \leq \nu_d(f)\Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^{-d}$$

for all $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$ and $x \in R(F)\backslash G(F)$.

Proof. The proof goes exactly the same as the Gan-Gross-Prasad model case in Proposition 7.1.1 of [B15]. We will postpone the proof to Appendix B. □

8.2 Proof of the Spectral Side

In this section, we are going to prove the spectral side of the trace formula.

Theorem 8.2.1. *For all $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$, we have*

$$I(f) = I_{spec}(f). \tag{8.2}$$

Here $I_{spec}(f)$ is defined in Section 5.2.

We follow the method developed by Beuzart-Plessis in [B15] for the GGP case. We fix $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$. For all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$, define

$$\begin{aligned}
K^A_{f,f'}(g_1,g_2) &= \int_{Z_G(F)\backslash G(F)} f(g_1^{-1}gg_2)f'(g)dg,\ g_1,g_2 \in G(F),\\
K^1_{f,f'}(g,x) &= \int_{Z_R(F)\backslash R(F)} K^A_{f,f'}(g,hx)\xi(h)\omega(h)dh,\ g,x \in G(F),\\
K^2_{f,f'}(x,y) &= \int_{Z_R(F)\backslash R(F)} K^1_{f,f'}(h^{-1}x,y)\xi(h)\omega(h)dh,\ x,y \in G(F),\\
J_{aux}(f,f') &= \int_{R(F)\backslash G(F)} K^2_{f,f'}(x,x)dx.
\end{aligned}$$

Proposition 8.2.2. 1. *The integral defining $K^A_{f,f'}(g_1, g_2)$ is absolutely convergent. For all $g_1 \in G(F)$, the map*

$$g_2 \in G(F) \to K^A_{f,f'}(g_1, g_2)$$

belongs to $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$. For all $d > 0$, there exists $d' > 0$ such that for all continuous semi-norm ν on $\mathcal{C}^w_{d'}(Z_G(F)\backslash G(F), \eta^{-1})$, there exists a continuous semi-norm μ on $\mathcal{C}(Z_G(F)\backslash G(F), \eta)$ such that

$$\nu(K^A_{f,f'}(g, \cdot)) \leq \mu(f')\Xi^G(g)\sigma_0(g)^{-d}$$

for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$ and $g \in G(F)$.

2. *The integral defining $K^1_{f,f'}(g, x)$ is absolutely convergent. For all $d > 0$, there exists $d' > 0$ and a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}(Z_G(F)\backslash G(F), \eta)$ such that*

$$|K^1_{f,f'}(g, x)| \leq \nu_{d,d'}(f')\Xi^G(g)\sigma_0(g)^{-d}\Xi^{R\backslash G}(x)\sigma_{R\backslash G}(x)^{d'}$$

for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$ and $g, x \in G(F)$.

3. *The integral defining $K^2_{f,f'}(x, y)$ is absolutely convergent. We have*

$$K^2_{f,f'}(x, y) = \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(y^{-1}))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi \qquad (8.3)$$

for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$ and $x, y \in G(F)$.

4. *The integral defining $J_{aux}(f, f')$ is absolutely convergent. And for all $d > 0$, there exists a continuous semi-norm ν_d on $\mathcal{C}(Z_G(F)\backslash G(F), \eta)$ such that $|K^2_{f,f'}(x, x)| \leq \nu_d(f')\Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^{-d}$ for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$ and $x \in R(F)\backslash G(F)$. Moreover, the linear map*

$$f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta) \to J_{aux}(f, f') \qquad (8.4)$$

is continuous.

Proof. (1) follows from Theorem 3.5.4(1). (2) follows from part (1) together with Lemma 4.3.1(2) and Lemma 6.2.3(2). For (3), the absolutely convergence follows from part (2) and Lemma 4.3.1(2). The equation (8.3) follows from Lemma 6.2.2(5).

For (4), by Lemma 6.2.2(1), the section

$$T(f'): \pi \in \Pi_{temp}(G,\eta) \mapsto \overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\pi(f) \in \mathrm{End}(\pi)^\infty$$

is smooth. It is also compactly supported since we are in the p-adic case. Then by the matrical Paley-Wiener Theorem, there exists a unique element $\varphi_{f'} \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ such that $\pi(\varphi_{f'}) = \overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\pi(f)$ for all $\pi \in \Pi_{temp}(G,\eta)$. Since f is strongly cuspidal, by Proposition 3.5.2, $\varphi_{f'}$ is also strongly cuspidal. Then by (8.3), we have

$$\begin{aligned} K^2_{f,f'}(x,x) &= \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x^{-1}))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi \\ &= \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(x)\pi(\varphi_{f'})\pi(x^{-1}))\mu(\pi)d\pi \\ &= \int_{Z_H(F)\backslash H(F)} \varphi_{f'}(x^{-1}hx)\xi(h)\omega(h)dh = I(\varphi_{f'},x). \end{aligned}$$

Here the third equation follows from Lemma 6.2.2(4). Then by Proposition 8.1.1, for all $d > 0$, there exists a continuous semi-norm ν_d on $\mathcal{C}(Z_G(F)\backslash G(F),\eta)$ such that $|K^2_{f,f'}(x,x)| \leq \nu_d(\varphi_{f'})\Xi^{G\backslash G}(x)^2\sigma_{G\backslash G}(x)^{-d}$ for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F),\eta)$ and $x \in R(F)\backslash G(F)$. Combining with Proposition 4.4.1(4), we know the integral defining $J_{aux}(f,f')$ is absolutely convergent. Finally, in order to prove the rest part of (4), it is enough to show that the map $\mathcal{C}(Z_G(F)\backslash G(F),\eta) \to \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1}) : f' \mapsto \varphi_{f'}$ is continuous. By the matrical Paley-Wiener Theorem, it is enough to show that the map

$$f' \in \mathcal{C}(Z_G(F)\backslash G(F),\eta) \mapsto (\pi \in \Pi_{temp}(G,\eta) \to \pi(\varphi_{f'}) = \overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\pi(f)) \in \mathcal{C}(\Pi_{temp}(G,\eta))$$

is continuous. This just follows from Lemma 6.2.2(1). This proves (4). □

Proposition 8.2.3. *For all $f' \in \mathcal{C}(Z_G(F)\backslash G(F),\eta)$, we have*

$$J_{aux}(f,f') = \int_{\Pi_{temp}(G,\chi)} \theta_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}d\pi.$$

Proof. The idea of proof comes from [B15]. Let $a : \mathbb{G}_m(F) \to Z_{G_0}(F)$ be a homomorphism defined by $a(t) = diag(t,t,1,1,t^{-1},t^{-1})$ in the split case, and $a(t) = diag(t,1,t^{-1})$ in the non-split case, then we know $\lambda(a(t)ha(t)^{-1}) = t\lambda(h)$ for all $h \in R(F), t \in \mathbb{G}_m(F)$. Fix $f' \in \mathcal{C}(Z_G(F)\backslash G(F),\eta)$, since we are in p-adic case, we can find an open compact neighborhood K_a of 1 in $F^\times$ such that $Ad_a(t)f' = f'$ for all $t \in K_a$. Let $\varphi \in C_c^\infty(F^\times)$ be the characteristic function on K_a divided by the measure of K_a. Then we have $f' = Ad_a(\varphi)(f')$ and $J_{aux}(f, f') = \int_{F^\times} \varphi(t) J_{aux}(f, Ad_a(t)f')dt$. By the definition of J_{aux}, we have

$$J_{aux}(f, f') = \int_{F^\times} \int_{R(F)\backslash G(F)} \varphi(t) K^2_{f, Ad_a(t)f'}(x, x)dxdt.$$

By part (4) of previous proposition, the double integral above is absolutely convergent. Then by changing variable $x \mapsto a(t)^{-1}x$ and switching the two integrals (note that the Jacobian of the map $h \in R(F) \mapsto a(t)ha(t)^{-1} \in R(F)$ is equal to $\delta_P(a(t))$), we have

$$J_{aux}(f, f') = \int_{R(F)\backslash G(F)} \int_{F^\times} \varphi(t)\delta_P(a(t))^{-1} K^2_{f, Ad_a(t)f'}(a(t)x, a(t)x)dtdx. \tag{8.5}$$

By the definition of $K^2_{f,f'}$, the inner integral is equal to

$$\int_{F^\times} \varphi(t)\delta_P(a(t))^{-1} \int_{Z_R(F)\backslash R(F)} K^1_{f, Ad_a(t)f'}(ha(t)x, a(t)x)\xi(h)^{-1}\omega(h)^{-1}dhdt.$$

By part (2) of previous proposition, the double integral above is still absolutely convergent. By changing variable $h \to a(t)^{-1}ha(t)$ and switching the two integrals, we have

$$\begin{aligned} & \int_{F^\times} \varphi(t)\delta_P(a(t))^{-1} K^2_{f, Ad_a(t)f'}(a(t)x, a(t)x)dt \qquad (8.6) \\ = & \int_{Z_R(F)\backslash R(F)} \int_{F^\times} \varphi(t) K^1_{f, Ad_a(t)f'}(a(t)hx, a(t)x)\psi(-t\lambda(h))\omega(h)^{-1}dtdh \\ = & \int_{Z_R(F)\backslash R(F)} \int_{F^\times} \varphi(t) K^1_{f, R_a(t)f'}(hx, a(t)x)\psi(-t\lambda(h))\omega(h)^{-1}dtdh. \end{aligned}$$

Here $R_a(t)$ stands for the right translation by $a(t)$. By the definition of $K^1_{f,f'}$, the inner integral above is equal to

$$\int_{F^\times} \varphi(t) \int_{Z_R(F)\backslash R(F)} K^A_{f, R_a(t)f'}(hx, h'a(t)x)\xi(h')\omega(h')dh'\psi(-t\lambda(h))\omega(h)^{-1}dt.$$

By changing variable $h' \to a(t)^{-1}h'a(t)h^{-1}$, this equals

$$\begin{aligned} & \int_{F^\times} \varphi(t) \int_{Z_R(F)\backslash R(F)} K^A_{f, R_a(t)f'}(hx, a(t)h'hx)\delta_P(a(t))\psi(t\lambda(h'))\omega(h')dh'dt \\ = & \int_{F^\times} \varphi(t) \int_{Z_R(F)\backslash R(F)} K^A_{f,f'}(hx, h'hx)\delta_P(a(t))\psi(t\lambda(h'))\omega(h')dh'dt. \end{aligned}$$

By part (1) of previous proposition, the integral above is absolutely convergent. By switching two integrals, we have

$$\begin{aligned} & \int_{F^\times} \varphi(t) K^1_{f,R_a(t)f'}(hx, a(t)x)\psi(-t\lambda(h))\omega(h)^{-1}dt \qquad (8.7) \\ = & \int_{Z_R(F)\backslash R(F)} \int_{F^\times} K^A_{f,f'}(hx, h'hx)\varphi(t)\delta_P(a(t))\psi(t\lambda(h'))\omega(h')dh'dt. \end{aligned}$$

We know $dt = |t|^{-1}d_at$ where d_at is an additive Haar measure on F. Let $\varphi'(t) = \varphi(t)\delta_P(a(t))|t|^{-1}$ and $\hat{\varphi}'(x) = \int_F \varphi'(t)\psi(tx)dt$ for $x \in F$. Combining (8.5), (8.6) and (8.7), we have

$$J_{aux}(f,f') = \int_{R(F)\backslash G(F)} \int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} K^A_{f,f'}(hx, h'hx)\hat{\varphi}'(\lambda(h'))\omega(h')dh'dhdx. \qquad (8.8)$$

For $N, M > 0$, let $\alpha_N : R(F)\backslash G(F) \to \{0,1\}$ (resp. $\beta_M : Z_G(F)\backslash G(F) \to \{0,1\}$) be the characteristic function of the set $\{x \in R(F)\backslash G(F) | \sigma_{R\backslash G}(x) \leq N\}$ (resp. $\{g \in Z_G(F)\backslash G(F) | \sigma_0(g) \leq M\}$). For $N \geq 1$ and $C > 0$, define

$$\begin{aligned} J_{aux,N}(f,f') &= \int_{R(F)\backslash G(F)} \alpha_N(x) \int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \\ & \quad K^A_{f,f'}(hx, h'hx)\hat{\varphi}'(\lambda(h'))\omega(h')dh'dhdx, \\ J_{aux,N,C}(f,f') &= \int_{R(F)\backslash G(F)} \alpha_N(x) \int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \\ & \quad \beta_{C\log(N)}(h') K^A_{f,f'}(hx, h'hx)\hat{\varphi}'(\lambda(h'))\omega(h')dh'dhdx. \end{aligned}$$

By equation (8.8), we have

$$J_{aux}(f,f') = \lim_{N\to\infty} J_{aux,N}(f'). \qquad (8.9)$$

We need to prove

(1) The triple integrals defining $J_{aux,N}(f,f')$ and $J_{aux,N,C}(f,f')$ are absolutely convergent. Moreover, there exists $C > 0$ such that

$$|J_{aux,N}(f,f') - J_{aux,N,C}(f,f')| \ll N^{-1}$$

for all $N \geq 1$.

In fact, since $\hat{\varphi}'$ is compactly supported on F, we have $|\hat{\varphi}'(\lambda)| \ll (1+|\lambda|)^{-1}$ for all $\lambda \in F$. Combining with Theorem 3.5.4, we know that there exists $d > 0$ such that

$$\begin{aligned} |J_{aux,N}(f,f')| &\ll \int_{R(F)\backslash G(F)} \alpha_N(x) \int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \\ &\quad \Xi^G(hx)\Xi^G(h'hx)\sigma_0(hx)^d\sigma_0(h'hx)^d(1+|\lambda(h')|)^{-1}dh'dhdx, \\ |J_{aux,N,C}(f,f')| &\ll \int_{R(F)\backslash G(F)} \alpha_N(x) \int_{Z_R(F)\backslash R(F)} \int_{Z_R(F)\backslash R(F)} \\ &\quad \beta_{C\log(N)}(h')\Xi^G(hx)\Xi^G(h'hx)\sigma_0(hx)^d\sigma_0(h'hx)^d(1+|\lambda(h')|)^{-1}dh'dhdx, \end{aligned}$$

and

$$|J_{aux,N}(f,f') - J_{aux,N,C}(f,f')| \ll \int_{R(F)\backslash G(F)} \alpha_N(x) \int_{Z_R(F)\backslash R(F)}$$

$$\times \int_{Z_R(F)\backslash R(F)} 1_{\sigma_0 \geq C\log(N)}(h')\Xi^G(hx)\Xi^G(h'hx)\sigma_0(hx)^d\sigma_0(h'hx)^d(1+|\lambda(h')|)^{-1}dh'dhdx.$$

for all $N \geq 1$ and $C \geq 1$. Applying (7) of Proposition 4.4.1 to the case $c = 1$, we know that there exists $d' > 0$ such that the first two integrals above are essentially bounded by

$$\int_{R(F)\backslash G(F)} \alpha_N(x)\Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^{d'}dx.$$

This is absolutely convergent since the integrand is compactly supported. Then applying (7) of Proposition 4.4.1 again, we know the third integral is essentially bounded by

$$e^{-\epsilon C\log(N)} \int_{R(F)\backslash G(F)} \alpha_N(x)\Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^{d'}dx,\ N \geq 1, C > 0.$$

for some $\epsilon, d' > 0$. By (4) of Proposition 4.4.1, there exists $d'' > 0$ such that the last integral is essentially bounded by $N^{d''}$ for all $N \geq 1$. Then once we choose C larger than $(d''+1)/\epsilon$, we have the estimation in (1). This proves (1).

From now on, we fix some $C > 0$ satisfies (1). Then we have

$$J_{aux}(f,f') = \lim_{N\to\infty} J_{aux,N,C}(f,f'). \tag{8.10}$$

Since the integral defining $J_{aux,N,C}$ is absolutely convergent, we can combine the first two parts and then switch two integrals, we have

$$\begin{aligned} J_{aux,N,C}(f,f') &= \int_{Z_G(F)\backslash G(F)} \alpha_N(g) \int_{Z_R(F)\backslash R(F)} \\ &\quad K^A_{f,f'}(g,h'g)\beta_{C\log(N)}(h')\hat{\varphi}'(\lambda(h'))\omega(h')dh'dg \qquad (8.11) \\ &= \int_{Z_R(F)\backslash R(F)} \beta_{C\log(N)}(h)\hat{\varphi}'(\lambda(h))\omega(h) \\ &\quad \times \int_{Z_G(F)\backslash G(F)} \alpha_N(g) K^A_{f,f'}(g,hg)dgdh. \end{aligned}$$

We are going to prove that for all $N \geq 1$, we have

$$|J_{aux,N,C}(f,f') - J_{aux,C}(f,f')| \ll N^{-1} \qquad (8.12)$$

where

$$J_{aux,C}(f,f') = \int_{Z_R(F)\backslash R(F)} \beta_{C\log(N)}(h)\hat{\varphi}'(\lambda(h))\omega(h) \int_{Z_G(F)\backslash G(F)} K^A_{f,f'}(g,hg)dgdh.$$

In fact, since f is strongly cuspidal, by Theorem 3.5.4(3), there exists $c_1 > 0$ such that for all $d > 0$, there exists $d' > 0$ such that

$$|K^A_{f,f'}(g,hg)| \ll \Xi^G(g)^2 \sigma_0(g)^{-d} e^{c_1\sigma_0(h)} \sigma_0(h)^{d'}$$

for all $g \in G(F)$ and $h \in R(F)$. Fix such $c_1 > 0$, and choose $d_0 > 0$ so that the function $g \to \Xi^G(g)^2\sigma_0(g)^{-d_0}$ is integrable on $G(F)/Z_G(F)$. Then for all $d > d_0$, there exists $d' > 0$ such that the left hand side of (8.12) is essentially bounded by

$$N^{c_1C-d+d_0} \log(N)^{d'} \int_{Z_R(F)\backslash R(F)} \beta_{C\log(N)}(h)dh$$

for all $N \geq 1$. It is easy to see that the integral above is essentially bounded by N^{c_2} for some $c_2 > 0$. Therefore once we choose $d > c_1C + d_0 + c_2 + 1$, we have the estimation in (8.12). This proves (8.12). Therefore we have

$$\begin{aligned} J_{aux}(f,f') &= \lim_{N\to\infty} \int_{Z_R(F)\backslash R(F)} \beta_{C\log(N)}(h)\hat{\varphi}'(\lambda(h))\omega(h) \\ &\quad \times \int_{Z_G(F)\backslash G(F)} K^A_{f,f'}(g,hg)dgdh. \qquad (8.13) \end{aligned}$$

Since f is strongly cuspidal, by Theorem 3.5.4(4), we have

$$\int_{Z_G(F)\backslash G(F)} K^A_{f,f'}(g,hg)dg = \int_{\Pi_{temp}(G,\eta)} \theta_f(\pi)\theta_{\bar{\pi}}(R(h^{-1})f')d\pi. \qquad (8.14)$$

Since π is tempered, $|\theta_{\bar{\pi}}(R(h^{-1})f')| \ll \Xi^G(h)$ for all $h \in R(F)$. Combining with the fact that $\theta_f(\pi)$ is smooth and compactly supported on $\Pi_{temp}(G,\eta)$, we have

$$\int_{\Pi_{temp}(G,\eta)} |\theta_f(\pi)\theta_{\bar{\pi}}(R(h^{-1})f')| d\pi \ll \Xi^G(h).$$

Combining with Lemma 4.3.1, we know that the integral

$$\int_{Z_R(F)\backslash R(F)} \hat{\varphi}'(\lambda(h))\omega(h) \int_{\Pi_{temp}(G,\eta)} \theta_f(\pi)\theta_{\bar{\pi}}(R(h^{-1})f') d\pi dh$$

is absolutely convergent. Combining with (8.13) and (8.14), the integral above is equal to $J_{aux}(f,f')$. Switching the two integrals and applying Lemma 6.1.2, we have

$$\begin{aligned} J_{aux}(f,f') &= \int_{\Pi_{temp}(G,\eta)} \theta_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{Ad_a(\varphi)f'}))} d\pi \\ &= \int_{\Pi_{temp}(G,\eta)} \theta_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} d\pi. \end{aligned}$$

This finishes the proof of the Proposition. $\square$

Now we are ready to prove Theorem 8.2.1. Recall that $I(f) = \int_{R(F)\backslash G(F)} I(f,x)dx$ where $I(f,x) = \int_{Z_R(F)\backslash R(F)} f(x^{-1}hx)\xi(h)\omega(h)dh$. Applying Lemma 6.2.2, we have

$$I(f,x) = \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x)^{-1})\mu(\pi)d\pi. \tag{8.15}$$

By Corollary 6.6.4, there exists a function $f' \in \mathcal{C}(Z_G(F)\backslash G(F),\eta)$ such that

$$\mathcal{L}_\pi(\pi(\overline{f'})) = m(\pi)$$

for all $\pi \in \Pi_{temp}(G,\eta)$ with $\pi(f) \neq 0$. Applying Theorem 6.2.1 and Corollary 6.6.4, for all $\pi \in \Pi_{temp}(G,\eta)$, $\mathcal{L}_\pi \neq 0$ if and only if $m(\pi) = 1$. Then (8.15) becomes

$$I(f,x) = \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x)^{-1})\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi.$$

Combining with Proposition 8.2.2(3), we have $I(f,x) = K^2_{f,f'}(x,x)$. Therefore $I(f) = J_{aux}(f,f')$. By the previous Proposition, together with the fact that $\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} = m(\pi) = m(\bar{\pi})$, we have

$$I(f) = J_{aux}(f,f') = \int_{\Pi_{temp}(G,\eta)} \theta_f(\pi)m(\bar{\pi})d\pi = I_{spec}(f).$$

This finishes the proof of Theorem 8.2.1.

Chapter 9

Localization

Starting from this chapter, we are going to prove the geometric side of the trace formula. As we proved in Proposition 5.2.3, it is enough to consider functions with trivial central character. We fix a strongly cuspidal function $f \in C_c^\infty(Z_G(F)\backslash G(F))$. In this chapter, our goal is to localize both sides of the trace formula in (5.7) (i.e $I_N(f)$ and $I_{geom}(f)$), which enables us to reduce the proof of the trace formula to the Lie algebra level.

In Section 9.1, we will talk about the localization at a semisimple element which is not conjugate to an element in $H(F)$. We can easily show that in this case, both $I_N(f)$ and $I_{geom}(f)$ are equal to zero. In Section 9.2, we consider the localization at the split elements of $H(F)$. By applying the spectral side of the trace formula and the inductional hypothesis, we can again show that both $I_N(f)$ and $I_{geom}(f)$ are equal to zero. In Section 9.3, we will talk about the localization of $I_N(f)$ at all other semisimple elements of $H(F)$. Finally in Section 9.4, we will talk about the localization of $I_{geom}(f)$.

9.1 A Trivial Case

If $x \in G_{ss}(F)$ that is not conjugate to an element in $H(F)$, then we can easily find a good neighborhood ω of 0 in $\mathfrak{g}_x(F)$ small enough such that $x\exp(X)$ is not conjugate to an element in $H(F)$ for any $X \in \omega$. Let $\Omega = Z_G(F)\cdot (x\exp(\omega))^G$. It follows that $\Omega \cap H(F) = \emptyset$. Suppose that f is supported on Ω. For every $t \in H_{ss}(F)$, the complement of Ω in $G(F)$ is an open neighborhood of t invariant under conjugation, and is away from the support of f. It follows that θ_f also vanishes on an open neighborhood of t, and hence that $I_{geom}(f) = 0$. On the other hand, the semisimple part of elements in $U(F)H(F)$ belongs to $H(F)$. Thus ${}^g f^{\xi} = 0$ for every $g \in G(F)$, and so $I_N(f) = 0$. Therefore the trace formula holds for f.

9.2 Localization at the split elements

If $x \in H_{ss}(F)$ such that $x = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a \neq b$. Note that this only happens in the split case, i.e. $G = \mathrm{GL}_6(F)$ and $H = \mathrm{GL}_2(F)$. We can easily find a good neighborhood ω of 0 in $\mathfrak{g}_x(F)$ small enough such that $x\exp(X)$ is not an elliptic element of G for any $X \in \omega$. Let $\Omega = Z_G(F) \cdot (x\exp(\omega))^G$. Then Ω does not contain any elliptic element of G. Suppose that f is supported on Ω. We are going to prove the trace formula for f, i.e.

$$\lim_{N\to\infty} I_N(f) = I_{geom}(f). \tag{9.1}$$

The main ingredients in our proof are the spectral expansion and the inductional hypothesis.

Firstly, by the spectral expansion we proved in the previous chapter, we have

$$\lim_{N\to\infty} I_N(f) = \int_{\Pi_{temp}(G,1)} \theta_f(\pi) m(\bar{\pi}) d\pi. \tag{9.2}$$

For any $\pi \in \Pi_{temp}(G,1)$, similar to the definition of $I_{geom}(f)$, we define the geometric multiplicity $m_{geom}(\pi)$ to be

$$m_{geom}(\pi) = \sum_{T\in\mathcal{T}} |W(H_0,T)|^{-1} \nu(T) \int_{Z_G(F)\backslash T(F)} c_\pi(t) D^H(t) \Delta(t) dt.$$

Here $c_\pi(t) = c_{\theta_\pi}(t)$ is the germ associated to the distribution character θ_π. Then by Proposition 3.5.3 together with the definition of $I_{geom}(f)$, we have

$$I_{geom}(f) = \int_{\Pi_{temp}(G,1)} \theta_f(\pi) m_{geom}(\bar{\pi}) d\pi. \tag{9.3}$$

Combining (9.2) and (9.3), we have

$$\lim_{N\to\infty} I_N(f) - I_{geom}(f) = \int_{\Pi_{temp}(G,1)} \theta_f(\pi)(m(\bar{\pi}) - m_{geom}(\bar{\pi})) d\pi. \tag{9.4}$$

Let $\Pi_2(G,1) \subset \Pi_{temp}(G,1)$ be the subset of discrete series, and let $\Pi'_{temp}(G,1) = \Pi_{temp}(G,1) - \Pi_2(G,1)$. For all $\pi \in \Pi_2(G,1)$, since the support of f does not contain any elliptic element, we have $\theta_f(\pi) = \mathrm{tr}(\pi(f)) = 0$. Therefore (9.4) becomes

$$\lim_{N\to\infty} I_N(f) - I_{geom}(f) = \int_{\Pi'_{temp}(G,1)} \theta_f(\pi)(m(\bar{\pi}) - m_{geom}(\bar{\pi}))d\pi. \tag{9.5}$$

For $\pi \in \Pi'_{temp}(G,1)$, we can find a proper parabolic subgroup $Q = LN$ and a discrete series τ of $L(F)$ such that $\pi = I_Q^G(\tau)$. By Corollary 6.6.4, we have $m(\pi) = m(\tau)$ where $m(\tau)$ is the multiplicity of the reduced model. Moreover, by inductional hypothesis as in Section 5.4, we have $m(\tau) = m_{geom}(\tau)$. Later in Lemma 13.1.1, we will also prove that $m_{geom}(\pi) = m_{geom}(\tau)$. Combining all the discussion above, we have

$$m(\pi) = m_{geom}(\pi)$$

for all $\pi \in \Pi'_{temp}(G,1)$. Put this equation into (9.5), we have

$$\lim_{N\to\infty} I_N(f) - I_{geom}(f) = 0.$$

This proves the trace formula.

9.3 Localization of $I_N(f)$

For $x \in H_{ss}(F)$, let $U_x = U \cap G_x$, fix a good neighborhood ω of 0 in $\mathfrak{g}_x(F)$, and let $\Omega = (x\exp(\omega))^G \cdot Z_G(F)$. By the discussion in the previous section, we can assume that x is elliptic in $H(F)$. We can decompose $\mathfrak{g}_{x,0}$ and $\mathfrak{h}_{x,0}$ into $\mathfrak{g}_{x,0} = \mathfrak{g}'_x \oplus \mathfrak{g}''$ and $\mathfrak{h}_{x,0} = \mathfrak{h}'_x \oplus \mathfrak{h}''$, where $\mathfrak{g}'_x = \mathfrak{h}'_x$ is the common center of $\mathfrak{g}_{x,0}$ and $\mathfrak{h}_{x,0}$, $\mathfrak{g}''$ and $\mathfrak{h}''$ are the semisimple parts. To be specific, the decomposition is given as follows: (Recall for any Lie algebra $\mathfrak{p}$, we define $\mathfrak{p}_0$ to be the subalgebra consisting of elements in $\mathfrak{p}$ with zero trace.)

- If x is contained in the center, then $G_x = G, H_x = H$. Define

$$\mathfrak{g}'_x = \mathfrak{h}'_x = 0, \mathfrak{g}'' = \mathfrak{g}_{x,0}, \mathfrak{h}'' = \mathfrak{h}_{x,0},$$

- If x is not split, then it is conjugate to a regular element in the torus T_v for some $v \in F^\times/(F^\times)^2$, $v \neq 1$. Recall T_v is the non-split torus of $H(F)$ that is

F-isomorphic to $F_v = F(\sqrt{v})$. In this case, $G_x = \mathrm{GL}_3(F_v), H_x = GL_1(F_v)$. Define

$$\begin{aligned}\mathfrak{g}'_x &= \mathfrak{h}'_x = \{\mathrm{diag}(a,a,a) \mid a \in F_v, \mathrm{tr}_{F_v/F}(a) = 0\},\\ \mathfrak{g}'' &= \mathfrak{sl}_3(F_v),\\ \mathfrak{h}'' &= 0.\end{aligned}$$

Then for every torus $T \in T(G_x)$ (here $T(G_x)$ stands for the set of maximal tori in G_x), we can write $\mathfrak{t}_0 = \mathfrak{t}' \oplus \mathfrak{t}''$ with $\mathfrak{t}' = \mathfrak{g}'_x = \mathfrak{h}'_x$. The idea of the decomposition above is that $\mathfrak{g}'_x = \mathfrak{h}'_x$ is the extra center in $\mathfrak{g}_x$, and $(\mathfrak{g}'', \mathfrak{h}'' \oplus \mathfrak{u}_x)$ stands for the reduced model after localization. In fact, if x is in the center, it is just the Ginzburg-Rallis model; when x is not in the center, it is the Whittaker model.

Remark 9.3.1. *There are two kinds of reduced models in our proof of the trace formula. In Section 4.5 and 5.4, we have already talked about the reduced models coming from the parabolic induction. Those reduced models have been used in the proof of the spectral side of the trace formula. Here we have another type of reduced models coming from localization. These models will used in the proof of the geometric side of the trace formula.*

From now on, we choose the function f such that $Supp(f) \subset \Omega$.

Definition 9.3.2. *Define a function ${}^g f_{x,\omega}$ on $\mathfrak{g}_{x,0}(F)$ by*

$$ {}^g f_{x,\omega}(X) = \begin{cases} f(g^{-1}x\exp(X)g), & \text{if } X \in \omega;\\ 0, & \text{otherwise.}\end{cases} \tag{9.6}$$

Here we still view ω as a subset of $\mathfrak{g}_{x,0}$ via the projection $\mathfrak{g}_x \to \mathfrak{g}_{x,0}$. We define

$$ {}^g f^{\xi}_{x,\omega}(X) = \int_{\mathfrak{u}_x(F)} {}^g f_{x,\omega}(X+N)\xi(N)dN, \tag{9.7}$$

$$ I_{x,\omega}(f,g) = \int_{\mathfrak{h}_{x,0}(F)} {}^g f^{\xi}_{x,\omega}(X)dX, \tag{9.8}$$

$$ I_{x,\omega,N}(f) = \int_{U_x(F)H_x(F)\backslash G(F)} I_{x,\omega}(f,g)\kappa_N(g)dg. \tag{9.9}$$

Remark 9.3.3. *The function $g \to I_{x,\omega}(f,g)$ is left $U_x(F)H_x(F)$-invariant. By Condition* (5) *of good neighborhood (as in Definition 3.1.1), there exists a subset $\Gamma \subset G(F)$, compact modulo center, such that ${}^g f_{x,\omega}(X) \equiv 0$ for $g \notin G_x(F)\Gamma$. Together with the fact that the function $g \to \kappa_N(g\gamma)$ on $G_x(F)$ has compact support modulo $U_x(F)H_x(F)$ for all $\gamma \in G(F)$, we know that the integrand in* (9.9) *is compactly supported. Therefore the integral is absolutely convergent.*

Proposition 9.3.4. *$I_N(f) = C(x)I_{x,\omega,N}(f)$ where $C(x) = D^H(x)\Delta(x)$.*

Proof. By the Weyl Integration Formula, we have

$$I(f,g) = \Sigma_{T\in T(H)} \mid W(H,T) \mid^{-1} \int_{Z_H(F)\backslash T(F)} J_H(t, {}^g f^{\xi}) D^H(t)^{1/2} dt \tag{9.10}$$

where

$$J_H(t,F) = D^H(t)^{1/2} \int_{H_t(F)\backslash H(F)} F(g^{-1}tg)dg$$

is the orbital integral. For given $T \in T(H)$ and $t \in T(F) \cap H_{reg}(F)$, we need the following lemma, the proof the lemma will be given after the proof this proposition.

Lemma 9.3.5. *For $t \in T(F)$, the followings hold.*

1. *If t does not belong to the following set*

$$\cup_{T_1\in T(H_x)} \cup_{w\in W(T_1,T)} w(x \exp(\mathfrak{t}_1(F) \cap \omega))w^{-1} \cdot Z_G(F),$$

then $J_H(t, {}^g f^{\xi}) = 0$. Here $W(T_1,T)$ is the set of isomorphisms between T and T_1 induced by conjugation by elements in $H(F)$, i.e. $W(T_1,T) = T\backslash\{h \in H(F) | hT_1h^{-1} = T\}/T_1$.

2. *If x is not contained in the center, each components in* (1) *are disjoint. If x is contained in the center, two components in* (1) *either are disjoint or coincide. They coincide if and only if $T = T_1$ in $T(H)$. Therefore, for each component (T_1, w), the number of components which coincide with it (include itself) is equal to $W(H_x, T_1)$.*

By the lemma above, we can rewrite the expression (9.10) of $I(f,g)$ as

$$I(f,g) = \Sigma_{T_1\in T(H_x)} \Sigma_{T\in T(H)} \Sigma_{w_1\in W(T_1,T)} |W(H,T)|^{-1} |W(H_x,T_1)|^{-1}$$

$$\times \int_{\mathfrak{t}_{1,0}(F)\cap\omega} J_H(w_1(w\exp(X))w_1^{-1}, {}^g f^\xi) D^H(w_1(w\exp(X))w_1^{-1})^{1/2} dX.$$

Note that both integrands above are invariant under $H(F)$-conjugate, $W(T_1, T) \neq \emptyset$ if and only if $T = T_1$ in $T(H)$, and in that case $W(T, T_1) = W(H, T)$. We have

$$I(f,g) = \Sigma_{T_1 \in T(H_x)} \mid W(H_x, T_1) \mid^{-1} \int_{\mathfrak{t}_{1,0}(F)\cap\omega} J_H(x\exp(X), {}^g f^\xi) D^H(x\exp(X))^{1/2} dX. \tag{9.11}$$

On the other hand, by Parts (3) and (5) of Proposition 3.1.2, for all $T_1 \in T(H_x)$ and for all $X \in \omega \cap \mathfrak{t}_{1,0,reg}(F)$, we have

$$\begin{aligned} J_H(x\exp(X), {}^g f^\xi) &= D^H(x\exp(X))^{1/2} \\ &\times \int_{H_x(F)\backslash H(F)} \int_{T_1(F)\backslash H_x(F)} {}^{yg} f^\xi(x\exp(h^{-1}Xh)) dh dy, \end{aligned} \tag{9.12}$$

and

$$D^H(x\exp(X)) = D^H(x) \cdot D^{H_x}(X). \tag{9.13}$$

So combining (9.11), (9.12), (9.13), together with the definition of $I_N(f)$ (as in (5.3)), we have

$$\begin{aligned} I_N(f) &= \int_{U(F)H(F)\backslash G(F)} \Sigma_{T_1 \in T(H_x)} \mid W(H_x, T_1) \mid^{-1} \\ &\times \int_{\mathfrak{t}_{1,0}(F)\cap\omega} J_H(x\exp(X), {}^g f^\xi) D^H(x\exp(X))^{1/2} dX \kappa_N(g) dg \qquad (9.14) \\ &= D^H(x) \int_{U(F)H_x(F)\backslash G(F)} \Phi(g) \kappa_N(g) dg \end{aligned}$$

where

$$\begin{aligned} \Phi(g) &= \Sigma_{T_1 \in T(H_x)} \mid W(H_x, T_1) \mid^{-1} \\ &\times \int_{\mathfrak{t}_{1,0}(F)\cap\omega} \int_{T_1(F)\backslash H_x(F)} {}^g f^\xi(x\exp(h^{-1}Xh)) dh D^{H_x}(X) dX. \end{aligned}$$

Applying the Weyl Integration Formula to $\Phi(g)$, we have

$$\Phi(g) = \int_{\mathfrak{h}_{x,0}(F)} \varphi_g(X) dX \tag{9.15}$$

where

$$\varphi_g(X) = \begin{cases} {}^g f^\xi(x\exp(X')), & \text{if } X = X' + Z, X' \in \omega, Z \in \mathfrak{z}_{\mathfrak{h}}(F); \\ 0, & \text{otherwise.} \end{cases} \tag{9.16}$$

On the other hand, for $X \in \omega \cap \mathfrak{h}_{x,reg}(F), g \in G(F)$,

$$\begin{aligned} {}^g f^{\xi}(x\exp(X)) &= \int_{U(F)} {}^g f(x\exp(X)u)\xi(u)du \\ &= \int_{U_x(F)\backslash U(F)} \int_{U_x(F)} {}^g f(x\exp(X)uv)\xi(uv)dudv. \end{aligned} \tag{9.17}$$

For $u \in U_x(F)$, the map $v \to (x\exp(X)u)^{-1}v^{-1}(x\exp(X)u)v$ is a bijection of $U_x(F)\backslash U(F)$. By the Condition $(7)_\rho$ of good neighborhood (as in Definition 3.1.1), the Jacobian of this map is

$$\mid \det((1-ad(x)^{-1}) \mid_{U(F)/U_x(F)}) \mid_F = \Delta(x).$$

Also it is easy to see that

$$\xi((x\exp(X)u)^{-1}v^{-1}(x\exp(X)u)v) = 1.$$

By making the transform $v \to (x\exp(X)u)^{-1}v^{-1}(x\exp(X)u)v$ in (9.17), we have

$$\begin{aligned} {}^g f^{\xi}(x\exp(X)) &= \Delta(x)\int_{U_x(F)\backslash U(F)} \int_{U_x(F)} {}^g f(v^{-1}x\exp(X)uv)\xi(u)dudv \\ &= \Delta(x)\int_{U_x(F)\backslash U(F)} \int_{U_x(F)} {}^{vg} f(x\exp(X)u)\xi(u)dudv. \end{aligned} \tag{9.18}$$

By Condition (6) of good neighborhood (as in Definition 3.1.1), for all $X \in \omega$, the map $\mathfrak{u}_x(F) \to U_x(F)$ given by

$$N \mapsto \exp(-X)\exp(X+N)$$

is a bijection and preserves the measure. Also we have

$$\xi(\exp(-X)\exp(X+N)) = \xi(N).$$

So we can rewrite (9.18) as

$${}^g f^{\xi}(x\exp(X)) = \Delta(x)\int_{U_x(F)\backslash U(F)} \int_{\mathfrak{u}_x(F)} {}^{vg} f(x\exp(X+N))\xi(N)dNdv.$$

For $X \in \omega_{reg}$, $X+N$ can be conjugated to X by an element in $G_x(F)$, so $X+N \in \omega$, and ${}^{vg}f(x\exp(X+N)) = {}^{vg}f_{x,\omega}(X+N)$ by the definition of ${}^g f_{x,\omega}$ (as in (9.6)). This implies that

$${}^g f^{\xi}(x\exp(X)) = \Delta(x)\int_{U_x(F)\backslash U(F)} {}^{vg} f^{\xi}_{x,\omega}(X)dv. \tag{9.19}$$

Now, combining (9.19) and (9.16), we have

$$\varphi_g(X) = \Delta(x) \int_{U_x(F)\backslash U(F)} {}^{vg} f^{\xi}_{x,\omega}(X') dv.$$

Then combining the above equation with (9.15) and changing the order of integration, we have

$$\Phi(g) = \Delta(x) \int_{U_x(F)\backslash U(F)} I_{x,\omega}(f, vg) dv. \tag{9.20}$$

Finally combining the above equation with (9.14) and using the fact that $C(x) = \Delta(x)D^H(x)$, we have

$$I_N(f) = C(x) \int_{U_x(F)H_x(F)\backslash G(F)} I_{x,\omega}(f, g)\kappa_N(g) dg = C(x) I_{x,\omega,N}(f).$$

This finishes the proof of the Proposition. □

Now we prove Lemma 9.3.5.

Proof. If $J_H(t, {}^g f^{\xi}) \neq 0$, there exists $u \in U(F)$ such that tu is conjugate to an element in $Supp(f)$. If we only consider the semisimple part, since we assume that $Supp(f) \subset \Omega = Z_G(F) \cdot (x \exp(\omega))^G$, there exist $y \in G(F), X \in \omega$ and $z \in Z_G(F)$, such that $yty^{-1} = x \exp(X) z$. By changing t to tz, we may assume that $z = 1$. Then by conjugating X by an element $y' \in G_x(F)$ and changing y to $y'y$, we may assume that $X \in \mathfrak{t}_1(F)$ for some $T_1 \in T(G_x)$.

If x is in the center, we have that $G_x = G$. Since $t \in H$, by changing y we may assume that $X \in \mathfrak{h} \cap \mathfrak{g}_x = \mathfrak{h}_x$. By further conjugating by an element in $H_x(F)$, we can just assume that $X \in \mathfrak{t}_1(F)$ for some $T_1 \in T(H_x)$. If x is not contained in the center, then $G_x = \mathrm{GL}_3(F_v)$. Assume that the eigenvalues of x are $\lambda, \lambda, \lambda, \mu, \mu, \mu$ for some $\lambda, \mu \in F_v, \lambda \neq \mu$. Note that for $t \in H$, its eigenvalues are of the same form, but may lie in some other quadratic extension of F. Now if ω is small enough with respect to $\mu - \lambda$, the eigenvalues of the given $X \in \omega$ must have the same form. It follows that $X \in \mathfrak{h}(F)$, and $X \in \mathfrak{h}(F) \cap \mathfrak{g}_x(F) = \mathfrak{h}_x(F)$. After a further conjugation by an element in $H_x(F)$, we can still assume that $X \in \mathfrak{t}_1(F)$ for some $T_1 \in T(H_x)$.

By the above discussion, we can always assume that $X \in \mathfrak{t}_1(F)$ for some $T_1 \in T(H_x)$. Since the Weyl group of G with respect to T equals the Weyl group of H with respect to T, any $G(F)-$conjugation between T and T_1 can be realized by an element in $H(F)$.

Here we define the Weyl group of T in G to be the quotient of the normalizer of T in G with the centralizer of T in G. Moreover, if such a conjugation exists, $T = T_1$ in $T(H)$ and the conjugation is given by the Weyl element $w \in W(T, T_1)$. This finishes the proof of Part (1).

Part (2) is very easy to verify. If x is not in the center, let λ and μ be the eigenvalues of x. Then $\lambda \neq \mu$, where λ and μ lie inside a quadratic extension of F. Once we choose ω small enough with respect to $\lambda - \mu$, it is easy to see that each components in (1) are disjoint. If x is in the center, by the proof of part (1), the components corresponding to T does not intersect with other components. Since the Weyl group $W(T_1, T) \simeq W(H, T)$ is of order 2, there are two components corresponding to T, and these two components coincide because ω is $G = G_x$-invariant in this case. This finishes the proof of (2). $\square$

9.4 Localization of $I_{geom}(f)$

We slightly modify the notation of Section 5.1: If $x \in Z_H(F)$, then $H_x = H$. In this case, we let $\mathcal{T}_x = \mathcal{T}$. (Recall that $\mathcal{T}$ is a subset of subtorus of H defined in Section 5.1.) If $x \notin Z_H(F)$, H_x is $\mathrm{GL}_1(F_v)$ for some $v \in F^\times/(F^\times)^2, v \neq 1$. Let $\mathcal{T}_x$ be the subset of $\mathcal{T}$ consisting of those nontrivial subtorus $T \in \mathcal{T}$ such that $T \in H_x$, i.e. $\mathcal{T}_x = \{T_v\}$. Now for $T \in \mathcal{T}_x$, we define the function $c_{f,x,\omega}$ on $\mathfrak{t}(F)$ as follows: It is zero for elements not contained in $\mathfrak{t}(F) \cap (\omega + \mathfrak{z}_{\mathfrak{g}}(F))$. For $X = X' + Y \in \mathfrak{t}(F)$ with $X' \in \omega, Y \in \mathfrak{z}_{\mathfrak{g}}(F)$, define

$$c_{f,x,\omega}(X) = c_f(x \exp(X')). \tag{9.21}$$

In fact, the function $\theta_{f,x,\omega}$ defined in (3.18) is a quasi-character in $\mathfrak{g}_x$, and the function $c_{f,x,\omega}$ we defined above is the germ associated to this quasi-character. Now we define the function Δ'' on $\mathfrak{h}_x(F)$ to be

$$\Delta''(X) = |\det(ad(X)\mid_{\mathfrak{u}_x(F)/(\mathfrak{u}_x(F))_X})|_F. \tag{9.22}$$

By Condition $(7)_\rho$ of Definition 3.1.1, we know that for every $X \in \omega$,

$$\Delta(x \exp(X)) = \Delta(x)\Delta''(X). \tag{9.23}$$

Let

$$I_{x,\omega}(f) = \Sigma_{T \in \mathcal{T}_x} |W(H_x, T)|^{-1} \nu(T) \int_{\mathfrak{t}_0(F)} c_{f,x,\omega}(X) D^{H_x}(X) \Delta''(X) dX. \tag{9.24}$$

By Proposition 5.1.2, the integral above is absolutely convergent.

Proposition 9.4.1. *With the notations above, we have*

$$I_{geom}(f) = C(x)I_{x,\omega}(f). \tag{9.25}$$

Proof. By applying the same argument as Lemma 9.3.5, we have the following properties for the function $c_f(t)$:

1. If $T \in \mathcal{T}$, and $t \in T(F)$, then $c_f(t) = 0$ if
$$t \notin \cup_{T_1 \in \mathcal{T}_x} \cup_{w \in W(T_1,T)} w(x\exp(\mathfrak{t}_1(F) \cup \omega))w^{-1} \cdot Z_G(F).$$

2. If x is not contained in the center, each components in (1) are disjoint. If x is contained in the center, two components in (1) either are disjoint or coincide. They coincide if and only if $T = T_1$ in $T(H)$. Therefore, for each component (T_1, w), the number of components which coincide with it (include itself) is equal to $W(H_x, T_1)$.

So we can rewrite the expression (5.6) of $I_{geom}(f)$ as

$$I_{geom}(f) = \Sigma_{T_1 \in \mathcal{T}_x} \Sigma_{T \in \mathcal{T}} \Sigma_{w_1 \in W(T_1,T)} \mid W(H,T) \mid^{-1} \mid W(H_x,T) \mid^{-1} \nu(T) \tag{9.26}$$
$$\times \int_{\mathfrak{t}_{1,0} \cap \omega} c_f(w_1(x\exp(X))w_1^{-1}) D^H(w_1(x\exp(X))w_1^{-1}) \Delta(x\exp(X))dX.$$

Since every integrand in (9.26) is invariant under $H(F)$-conjugation, together with Proposition 3.1.2(5) and (9.23), we have

$$D^H(x\exp(X))\Delta(x\exp(X)) = D^H(x)D^{H_x}(X)\Delta(x)\Delta''(X).$$

Then (9.26) becomes

$$\begin{aligned} I_{geom}(f) &= D^H(x)\Delta(x)\Sigma_{T_1 \in \mathcal{T}_x} \nu(T_1) \mid W(H_x,T) \mid^{-1} \\ &\quad \times \int_{\mathfrak{t}_{1,0}(F)} c_{f,x,\omega}(X) D^{H_x}(X)\Delta''(X)dX \\ &= C(x)I_{x,\omega}(f). \end{aligned}$$

This finishes the proof of the Proposition. $\square$

Chapter 10

Integral Transfer

10.1 The Problem

In this section, let (G', H', U') be one of the following:

1. $G' = \mathrm{GL}_6(F), H' = \mathrm{GL}_2(F)$, U' is the unipotent radical of the parabolic subgroup whose Levi is $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$.

2. $G' = \mathrm{GL}_3(D), H' = \mathrm{GL}_1(D)$, U' is the unipotent radical of the parabolic subgroup whose Levi is $\mathrm{GL}_1(D) \times \mathrm{GL}_1(D) \times \mathrm{GL}_1(D)$.

3. $G' = \mathrm{GL}_3(F_v), H' = \mathrm{GL}_1(F_v)$, for some $v \in F^\times/(F^\times)^2$ with $v \neq 1$, U' is the unipotent radical of the Borel subgroup.

This basically means that (G', H', U') is of the form (G_x, H_x, U_x) for some elliptic element $x \in H_{ss}(F)$. Our goal is to simplify the integral $I_{x,\omega,N}(f)$ defined in last section. To be specific, in the definition of $I_{x,\omega,N}(f)$, we first integrate over the Lie algebra of $H_x U_x$, then integrate over $U_x H_x \backslash G_x$. In this section, we are going to transfer this integral into the form $\int_{\mathfrak{t}^0(F)} \int_{A_T(F)\backslash G(F)}$ where T runs over maximal torus in G_x and $\mathfrak{t}^0(F)$ is a subset of $\mathfrak{t}(F)$ which will be defined later. The reason for doing this is that we want to apply Arthur's local trace formula which is of the form $\int_{A_T(F)\backslash G(F)}$. Our method is to study the orbit of the slice representation. We will only write down the proof for the first two situations. The proof for the last situation follows from the same, but easier arguments, and hence we will skip the proof here. So we will still use (G, H, U) instead of (G', H', U') in this section. We fix a truncated function $\kappa \in C_c^\infty(U(F)H(F)\backslash G(F))$, and a function $f \in C_c^\infty(\mathfrak{g}_0(F))$. Recall that in Section 5.3, we have defined

$$f^\xi(Y) = \int_{\mathfrak{u}(F)} f(Y+N)\xi(N)dN$$

and

$$I(f,g) = \int_{\mathfrak{h}_0(F)} {}^g f^\xi(Y)dY.$$

Let

$$I_\kappa(f) = \int_{U(F)H(F)\backslash G(F)} I(f,g)\kappa(g)dg. \tag{10.1}$$

We are going to study $I_\kappa(f)$.

10.2 Premier Transform

For $\Xi = \begin{pmatrix} 0 & 0 & 0 \\ aI_2 & 0 & 0 \\ 0 & bI_2 & 0 \end{pmatrix}$, we have that $\xi(N) = \psi(<\Xi, N>)$ for $N \in \mathfrak{u}(F)$. Here we use I_2 to denote the identity element in $\mathfrak{h}(F)$, i.e. in split case, I_2 is the two by two identity matrix; and in nonsplit case, I_2 is the identity element in the quaternion algebra. Define

$$\Lambda_0 = \{\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \mid A+B+C=0\}$$

and

$$\Sigma = \Lambda_0 + \mathfrak{u}.$$

Lemma 10.2.1. *For all $f \in C_c^\infty(\mathfrak{g}_0(F))$ and $Y \in \mathfrak{h}_0(F)$, we have*

$$(f^\xi)\hat{}(Y) = \int_\Sigma \hat{f}(\Xi + Y + X)dX.$$

Proof. Since $\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{h}_0 \oplus \Lambda_0 \oplus \mathfrak{u}$, we may assume that $f = f_{\bar{\mathfrak{u}}} \otimes f_{\mathfrak{h}_0} \otimes f_{\Lambda_0} \otimes f_{\mathfrak{u}}$. Then we have

$$\begin{aligned} \hat{f} &= \hat{f}_{\bar{\mathfrak{u}}} \otimes \hat{f_{\mathfrak{h}_0}} \otimes \hat{f_{\Lambda_0}} \otimes \hat{f}_{\mathfrak{u}}, \\ f^\xi(Y) &= f_{\bar{\mathfrak{u}}}(0) \otimes f_{\mathfrak{h}_0}(Y) \otimes f_{\Lambda_0}(0) \otimes \hat{f}_{\mathfrak{u}}(\Xi), \\ (f^\xi)\hat{}(Y) &= f_{\bar{\mathfrak{u}}}(0) \otimes \hat{f_{\mathfrak{h}_0}}(Y) \otimes f_{\Lambda_0}(0) \otimes \hat{f}_{\mathfrak{u}}(\Xi). \end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\Sigma} \hat{f}(\Xi + Y + X)dX &= \hat{f}_{\mathfrak{u}}(\Xi)\hat{f_{\mathfrak{h}_0}}(Y)\int_{\Sigma} \hat{f_{\Lambda_0}} \otimes \hat{f}_{\bar{\mathfrak{u}}}(X)dX \\ &= f_{\bar{\mathfrak{u}}}(0) \otimes \hat{f_{\mathfrak{h}_0}}(Y) \otimes f_{\Lambda_0}(0) \otimes \hat{f}_{\mathfrak{u}}(\Xi).\end{aligned}$$

This finishes the proof of the Lemma. □

10.3 Description of the Affine Space $\Xi + \Sigma$

Let $\Lambda = \{\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\}$ be a subset of $\mathfrak{u}(F)$.

Lemma 10.3.1. $\Xi + \Sigma$ *is stable under the* $U(F)$*-conjugation. The map*

$$U(F) \times (\Xi + \Lambda) \to \Xi + \Sigma : \ (u, x) \mapsto u^{-1}Xu \tag{10.2}$$

is an isomorphism of algebraic varieties.

Proof. We have the following two equations

$$\begin{pmatrix} I_2 & X & Z \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ aI_2 & 0 & 0 \\ 0 & bI_2 & 0 \end{pmatrix} \begin{pmatrix} I_2 & -X & XY - Z \\ 0 & I_2 & -Y \\ 0 & 0 & I_2 \end{pmatrix}$$

$$= \begin{pmatrix} aX & bZ - X^2 & aX^2Y - aXZ - bYZ \\ aI_2 & bY - aX & aXY - aZ - bY^2 \\ 0 & bI_2 & -bY \end{pmatrix},$$

and

$$\begin{pmatrix} I_2 & X & Z \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_2 & -X & XY - Z \\ 0 & I_2 & -Y \\ 0 & 0 & I_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & B + XC \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the map (10.2) is clearly injective. On the other hand, for any element in $\Xi + \Sigma$, applying the first equation above, we can choose X and Y to match the elements in the diagonal. Then applying the second equation, we can choose Z to match the element in the first row second column. Finally applying second equation again, we can choose B and C to match the elements in the first row third column and in the second row third column. Therefore the map (10.2) is surjective.

Now we have proved the map (10.2) is a bijection of points. In order to show it is an isomorphism of algebraic varieties, we only need to find the inverse map. Let $\begin{pmatrix} A' & T_1 & T_2 \\ aI_2 & B' & T_3 \\ 0 & bI_2 & C' \end{pmatrix}$ be an element in $\Xi + \Sigma$. Set

$$X = \frac{1}{a}A',\ Y = -\frac{1}{b}C',\ Z = \frac{T_1 + X^2}{b}, \tag{10.3}$$

$$C = T_3 - aXY + aZ + bY^2,\ B = T_2 - aX^2Y + aXZ + bYZ - XC,$$

then by the two equations above, we have

$$\begin{pmatrix} I_2 & X & Z \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & B \\ aI_2 & 0 & C \\ 0 & bI_2 & 0 \end{pmatrix} \begin{pmatrix} I_2 & -X & XY - Z \\ 0 & I_2 & -Y \\ 0 & 0 & I_2 \end{pmatrix} = \begin{pmatrix} A' & T_1 & T_2 \\ aI_2 & B' & T_3 \\ 0 & bI_2 & C' \end{pmatrix}.$$

Therefore the map (10.3) is the inverse map of (10.2), also it is clearly algebraic. This finishes the proof of the Lemma. □

Definition 10.3.2. *We say an element $W \in \Xi + \Sigma$ is in "generic position" if it satisfies the following two conditions:*

1. *W is semisimple regular.*
2. *W is conjugated to an element $\begin{pmatrix} 0 & 0 & X \\ aI_2 & 0 & Y \\ 0 & bI_2 & 0 \end{pmatrix} \in \Sigma + \Lambda$ such that X, Y are semisimple regular and $XY - YX$ is not nilpotent. In particular, this implies $H_X \cap H_Y = Z_H$.*

Let $\Xi + \Sigma^0$ be the subset of $\Xi + \Sigma$ consisting of elements in "generic position". It is a Zariski open subset of $\Xi + \Sigma$. Let $\Xi + \Lambda^0 = (\Xi + \Sigma^0) \cap (\Xi + \Lambda)$.

10.4 Orbits in $\Xi + \Lambda^0$

Lemma 10.4.1. *The group $Z_G(F)\backslash H(F)U(F)$ acts by conjugation on $\Xi+\Sigma^0$, and this action is free. Two elements in $\Xi+\Sigma^0$ are conjugated to each other in $G(F)$ if and only if they are conjugated to each other by an element in $H(F)U(F)$.*

Proof. For the first part, by Lemma 10.3.1, we only need to show that the action of $Z_G(F)\backslash H(F)$ on $\Xi+\Lambda^0$ is free. This just follows from the "generic position" assumption.

For the second part, given $x, y \in \Xi + \Sigma^0$, which are conjugated to each other by an element in $G(F)$. By conjugating both elements by some elements in $U(F)$, we may assume that $x, y \in \Xi + \Lambda^0$. Let

$$x = \begin{pmatrix} 0 & 0 & X_1 \\ aI_2 & 0 & X_2 \\ 0 & bI_2 & 0 \end{pmatrix}, \ y = \begin{pmatrix} 0 & 0 & Y_1 \\ aI_2 & 0 & Y_2 \\ 0 & bI_2 & 0 \end{pmatrix}.$$

We only need to find $h \in H(F)$ such that $h^{-1}X_i h = Y_i$ for $i = 1, 2$. The characteristic polynomial of x is

$$\det(x - \lambda I_6) = \det(\begin{pmatrix} -\lambda I_2 & 0 & X_1 \\ aI_2 & -\lambda I_2 & X_2 \\ 0 & bI_2 & -\lambda I_2 \end{pmatrix}),$$

which can be calculated as follows:

$$\begin{aligned} \det(x - \lambda I_6) &= \det(\begin{pmatrix} 0 & -\lambda^2/aI_2 & X_1 + \lambda/aX_2 \\ aI_2 & -\lambda I_2 & X_2 \\ 0 & bI_2 & -\lambda I_2 \end{pmatrix}) \\ &= a^2 \cdot \det(\begin{pmatrix} -\lambda^2/aI_2 & X_1 + \lambda/aX_2 \\ bI_2 & -\lambda I_2 \end{pmatrix}) \\ &= a^2 \cdot \det(\begin{pmatrix} 0 & X_1 + \lambda/aX_2 - \frac{\lambda^3}{ab}I_2 \\ bI_2 & -\lambda I_2 \end{pmatrix}). \end{aligned}$$

Hence we have

$$\det(x - \lambda I_6) = a^2b^2 \det(X_1 + \lambda/aX_2 - \frac{\lambda^3}{ab}I_2).$$

Therefore, up to some sign constants ± 1, the coefficients of the characteristic polynomial of x are determined by some data of X_1, X_2 given as follows:

$$\text{coefficient of } \lambda^4 = b\mathrm{tr}(X_2), \tag{10.4}$$

$$\text{coefficient of } \lambda^3 = ab\mathrm{tr}(X_1), \tag{10.5}$$

$$\text{coefficient of } \lambda^2 = b^2 \det(X_2), \tag{10.6}$$

$$\text{coefficient of } \lambda = ab^2(\lambda - \text{coefficient of } \det(X_1 + \lambda X_2)), \tag{10.7}$$

and

$$\text{coefficient of } \lambda^0 = a^2 b^2 \det(X_1). \tag{10.8}$$

Here the equation holds up to ± 1 which will not affect our later calculation. **Note that in the nonsplit case, the determinant means the composition of the determinant of the matrix and the norm of the quaternion algebra; and the trace means the composition of the trace of the matrix and the trace of the quaternion algebra.**

We can have the same results for y. Now if x and y are conjugated to each other by element in $G(F)$, their characteristic polynomials are equal. Hence we have

$$\mathrm{tr}(X_2) = \mathrm{tr}(Y_2), \tag{10.9}$$

$$\mathrm{tr}(X_1) = \mathrm{tr}(Y_1), \tag{10.10}$$

$$\det(X_2) = \det(Y_2), \tag{10.11}$$

$$\lambda - \text{coefficient of } \det(X_1 + \lambda X_2) = \lambda - \text{coefficient of } \det(Y_1 + \lambda Y_2), \tag{10.12}$$

and

$$\det(X_1) = \det(Y_1). \tag{10.13}$$

By the "generic positive" assumption, X_i and Y_i are semisimple regular. Then the above equations tell us that X_i and Y_i are conjugated to each other by some elements in $H(F)$ (i=1,2).

We first deal with the split case, i.e. $G = GL_6(F)$ **and** $H = GL_2(F)$**.** By further conjugating by some elements in $H(F)$, we may assume that $X_1 = Y_1$ be one of the following forms:

$$X_1 = Y_1 = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix};\ X_1 = Y_1 = \begin{pmatrix} s & tv \\ t & s \end{pmatrix}$$

where $v \in F^\times/(F^\times)^2, v \neq 1$. By the "generic positive" assumption, if we are in the first case, $s \neq t$; and if we are in the second case, $t \neq 0$. Let

$$X_2 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y_2 = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Case 1: If $X_1 = Y_1 = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ with $s \neq t$. By (10.12), we have $sx_{22} + tx_{11} = sy_{22} + ty_{11}$. Combining this with (10.9), we have $x_{11} = y_{11}$ and $x_{22} = y_{22}$. By applying (10.11), we have $x_{12}x_{21} = y_{12}y_{21}$. By the "generic position" assumption, $x_{12}x_{21}y_{12}y_{21} \neq 0$, and hence $\frac{x_{12}}{y_{12}} = \frac{y_{21}}{x_{21}}$. So we can conjugate X_2 by an element of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ to get Y_2. Therefore we can conjugate X_1, X_2 to Y_1, Y_2 simultaneously via an element in $H(F)$.

Case 2: If $X_1 = Y_1 = \begin{pmatrix} s & tv \\ t & s \end{pmatrix}$ with $t \neq 0$. By (10.12), we have $s\mathrm{tr}(X_2) - t(vx_{21} + x_{12}) = s\mathrm{tr}(Y_2) - t(vy_{21} + y_{12})$. Combining with (10.9), we have $vx_{21} + x_{12} = vy_{21} + y_{12}$. Let

$$x_{11} + x_{22} = y_{11} + y_{22} = A,$$

$$x_{11}x_{22} - x_{12}x_{21} = y_{11}y_{22} - y_{12}y_{21} = B,$$

and

$$vx_{21} + x_{12} = vy_{21} + y_{12} = C.$$

By the first and third equations, we can replace x_{12}, x_{22} by x_{21}, x_{11} in the second equation. We can do the same thing for the y's. It follows that

$$Ax_{11} - x_{11}^2 - Cx_{21} + vx_{21}^2 = Ay_{11} - y_{11}^2 - Cy_{21} + vy_{21}^2 = B. \tag{10.14}$$

Now for all $k \in F$, we have

$$\begin{pmatrix} k & v \\ 1 & k \end{pmatrix} x \begin{pmatrix} k & v \\ 1 & k \end{pmatrix}^{-1}$$

$$= \frac{1}{k^2 - v} \begin{pmatrix} k^2 x_{11} + kvx_{21} - kx_{12} - vx_{22} & k^2 x_{12} + kvx_{22} - kvx_{11} - v^2 x_{21} \\ kx_{11} + k^2 x_{21} - x_{12} - kx_{22} & kx_{12} + k^2 x_{22} - vx_{11} - kvx_{21} \end{pmatrix}.$$

If we write the above action in terms of x_{11}, x_{21}, we have

$$\begin{aligned} x_{11} &\mapsto (x_{11}k^2 + (2vx_{21} - C)k + vx_{11} - vA)/(k^2 - v) := k.x_{11}, \\ x_{21} &\mapsto (x_{21}k^2 + (2x_{11} - A)k + vx_{21} - C)/(k^2 - v) := k.x_{21}. \end{aligned}$$

If we want $y_{21} = k.x_{21}$, we need

$$((x_{21} - y_{21})k^2 + (2x_{11} - A)k + vx_{21} - C + vy_{21}) = 0. \tag{10.15}$$

The discriminant of (10.15) is equal to

$$\begin{aligned} \Delta \text{ of } (10.15) &= 4x_{11}^2 - 4Ax_{11} + A^2 - 4v(x_{21}^2 - y_{21}^2) + 4C(x_{21} - y_{21}) \\ &= A^2 - 4B + 4vy_{21}^2 - 4Cy_{21} \\ &= \Delta \text{ of } (10.14), \end{aligned}$$

where the second equality comes from (10.14). So the discriminant of (10.15) is a square in F. Hence we can find some $k \in F$ such that $y_{21} = k.x_{21}$. By conjugating by element of the form $\begin{pmatrix} k & v \\ 1 & k \end{pmatrix}$, we may assume that $x_{21} = y_{21}$. This also implies $x_{12} = y_{12}$. Then by (10.11) and (10.9), we have $x_{11} = y_{11}, x_{22} = y_{22}$ or $x_{11} = y_{22}, x_{22} = y_{11}$.

If $x_{11} = x_{22}$, we are done. If $x_{11} \neq x_{22}$, the discriminant of (10.14) is nonzero, so (10.15) also has nonzero discriminant. Therefore, it have two solutions k_1, k_2. Both k_1 and k_2 will make $x_{12} = y_{12}, x_{21} = y_{21}$. By the "generic positive" assumption, k_1, k_2 conjugate x to different elements. So one of them will conjugate x to y. Therefore we have proved that we can conjugate X_1, X_2 to Y_1, Y_2 simultaneously via an element in $H(F)$.

We now deal with the non-split case. We can just use the same argument as in Case 2. The calculation is very similar, and the details will be omitted here. This finishes the proof of the Lemma. □

Remark 10.4.2. *As pointed out by a referee, there is another way to prove Case 2 by extension of scalars. Let E/F be a finite Galois extension such that X_1 is split over E. Then by the argument in Case 1, we can find an element $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $H(E)$ conjugating X_1, X_2 to Y_1, Y_2. Without loss of generality, we may assume that $a \neq 0$. Also up to an element in $Z_H(E)$, we may assume that $a = 1$. For any $\tau \in Gal(E/F)$, $\tau(h)$ will also conjugate X_1, X_2 to Y_1, Y_2. By the generic position assumption, $\tau(h) = hz$ for some $z \in Z_H(E)$. But since $1 = a = \tau(a)$, z must be the identity element which implies that $h = \tau(h)$. Therefore $h \in H(F)$, and this proves Case 2. The same argument can be also applied to the non-split case.*

Remark 10.4.3. *To summarize, we have an injective analytic morphism*

$$(\Xi + \Sigma_0)/H(F)U(F) \longrightarrow \coprod_{T \in T(G)} \mathfrak{t}(F)/W(G,T). \tag{10.16}$$

For each $T \in T(G)$, let $\mathfrak{t}^0(F)/W(G,T)$ be the image of the map above. Then it is easy to prove the following statements.

1. *$\mathfrak{t}^0(F) \subset \mathfrak{t}_0(F)$. Recall that $\mathfrak{t}_0(F)$ is the subset of $\mathfrak{t}(F)$ consisting of the elements with zero trace.*
2. *$\mathfrak{t}^0(F)$ is invariant under scalar in the sense that for all $t \in \mathfrak{t}^0(F)$ and $\lambda \in F^\times$, we have $\lambda t \in \mathfrak{t}^0(F)$.*
3. *$\mathfrak{t}^0(F)$ is an open subset of $\mathfrak{t}_0(F)$ under the topology on $\mathfrak{t}_0(F)$ as an F-vector space.*
4. *If T is split which is only possible when $G = GL_6(F)$, then $\mathfrak{t}^0(F) = \mathfrak{t}_{0,reg}(F)$. (This will be proved in the proof of Lemma 11.5.1).*

As a result, we have a bijection

$$(\Xi + \Sigma_0)/H(F)U(F) \longrightarrow \coprod_{T \in T(G)} \mathfrak{t}^0(F)/W(G,T). \tag{10.17}$$

Now we study the change of measures under the map (10.17) (i.e. the Jacobian). We fix selfdual measures on $\Xi + \Sigma_0$ and $H(F)U(F)$, this induces a measure on the quotient which gives a measure $d_1 t$ on $\mathfrak{t}^0(F)/W(G,T)$ via the bijection (10.17) for any $T \in T(G)$. On the other hand, we also have a selfdual measure dt on $\mathfrak{t}^0(F)/W(G,T)$. The following lemma tells us the relations between $d_1 t$ and dt.

Lemma 10.4.4. *For any $T \in T(G)$, $d_1 t = D^G(t)^{1/2} dt$ for all $t \in \mathfrak{t}^0(F)$.*

Proof. Let $d_2 t$ be the measure on $\mathfrak{t}^0(F)/W(G,T)$ coming from the quotient $\Xi{+}\Lambda^0/H(F)$. By Lemma 10.3.1,

$$d_2 t = a^4 b^8 d_1 t. \tag{10.18}$$

For $T_H \in T(H)$, define $\Xi{+}T_H = \{\Lambda(X_1, X_2) = \begin{pmatrix} 0 & 0 & X_1 \\ aI_2 & 0 & X_2 \\ 0 & bI_2 & 0 \end{pmatrix} \in \Xi{+}\Lambda^0 | X_1 \in \mathfrak{t}_H(F)\}$.

Then the bijection

$$\Xi + \Lambda^0/H(F) \to \coprod_{T \in T(G)} \mathfrak{t}^0(F)/W(G,T)$$

factors through

$$\Xi + \Lambda^0/H(F) \to \coprod_{T_H \in T(H)} \Xi + T_H/T_H(F) \to \coprod_{T \in T(G)} \mathfrak{t}^0(F)/W(G,T).$$

By the Weyl Integration Formula, the Jacobian of the first map is $D^H(X_1)^{-1}$ at $\Lambda(X_1, X_2)$. Combining with (10.18), we only need to show that the Jacobian of the map

$$\coprod_{T_H \in T(H)} \Xi + T_H/T_H(F) \to \coprod_{T \in T(G)} \mathfrak{t}^0(F)/W(G,T)$$

is $a^4 b^8 D^H(X_1) D^G(\Lambda(X_1, X_2))^{-1/2}$ at $\Lambda(X_1, X_2)$. We consider the composite map

$$\Xi + T_H/T_H(F) \to \coprod_{T \in T(G)} \mathfrak{t}^0(F)/W(G,T) \to F^5 \tag{10.19}$$

where the second map is taking the coefficients of the characteristic polynomial. (since the trace is always 0, we only take the coefficients from degree 0 to 4.) As the Jacobian of the second map is $D^G(t)^{1/2}$ at $t \in \mathfrak{t}^0(F)$, we only need to show that the Jacobian of the composite map (10.19) is $a^4 b^8 D^H(X_1)$ at $\Lambda(X_1, X_2)$.

We only write down the proof for the case when T_H is split, the proof for the rest cases is similar. If T_H is split, we may assume that $T_H = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\}$. By the generic position assumption, we know

$$\Xi + T_H/T_H(F) = \{\Lambda(X_1, X_2) | X_1 = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix},\ X_2 = \begin{pmatrix} m_1 & 1 \\ x & n_1 \end{pmatrix},\ m \neq n,\ x \neq 0\}.$$

The measure on $\Xi + T_H/T_H(F)$ is just $dmdndm_1dn_1dx$. Note that we always use the selfdual measure on F. In the proof of Lemma 10.4.1, we have written down the map (10.19) explicitly (i.e (10.4) to (10.8)):

$$(m, n, m_1, n_1, x) \mapsto (b(m_1 + n_1), ab(m + n), b^2(m_1n_1 - x), ab^2(mn_1 + m_1n), a^2b^2mn). \tag{10.20}$$

By a simple computation, the Jacobian of (10.20) is

$$a^4b^8|(m-n)^2|_F = a^4b^8D^H(X_1).$$

This finishes the proof of the lemma. □

10.5 Local Sections

For $T \in T(G)$, we can fix a locally analytic map

$$\mathfrak{t}^0(F) \to \Xi + \Sigma^0 : Y \to Y_\Sigma \tag{10.21}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \Xi + \Sigma^0 & \longrightarrow & \mathfrak{t}^0(F)/W(G,T) \\ \nwarrow & & \nearrow \\ & \mathfrak{t}^0(F) & \end{array}$$

Then we can also find a map $Y \to \gamma_Y$ such that $Y_\Sigma = \gamma_Y^{-1} Y \gamma_Y$.

Lemma 10.5.1. *If ω_T is a compact subset of $\mathfrak{t}_0(F)$, we can choose the map $Y \to Y_\Sigma$ such that the image of $\mathfrak{t}^0(F) \cap \omega_T$ is contained in a compact subset of $\Xi + \Lambda$.*

Proof. We only write down the proof for the split case, the proof for the non-split case is similar. Given $t \in \mathfrak{t}^0(F)$, we want to find an element of the form $\begin{pmatrix} 0 & 0 & X \\ aI_2 & 0 & Y \\ 0 & bI_2 & 0 \end{pmatrix}$ that is a conjugation of t. As in the proof of Lemma 10.4.1, the characteristic polynomial of t gives us the determinant and trace of both X and Y, and also an extra equation (i.e. the

λ-coefficient). Once t lies in a compact subset, all these five values are bounded. Hence we can definitely choose X and Y such that their coordinates are bounded. Therefore, both elements belong to a compact subset. $\square$

Combining the above Lemma and Proposition 2.4.2, we can choose the map $Y \to \gamma_Y$ with the property that there exists $c > 0$ such that

$$\sigma(\gamma_Y) \leq c(1+ \mid \log D^G(Y) \mid) \tag{10.22}$$

for all $Y \in \mathfrak{t}^0(F) \cap \omega_T$.

10.6 Calculation of $I_\kappa(f)$

By Lemma 10.2.1,

$$I(f,g) = ({}^g f^\xi)\hat{}(0) = \int_\Sigma {}^g\hat{f}(\Xi + X)dX.$$

This implies

$$I_\kappa(f) = \int_{H(F)U(F)\backslash G(F)} \int_\Sigma {}^g\hat{f}(\Xi + X)dX\kappa(g)dg.$$

By Lemma 10.4.1, Remark 10.4.3 and Lemma 10.4.4, the interior integral equals

$$\Sigma_{T\in T(G)} \mid W(G,T) \mid^{-1} \int_{Z_H(F)\backslash H(F)U(F)} \int_{\mathfrak{t}^0(F)} {}^g\hat{f}(y^{-1}\gamma_Y^{-1}Y\gamma_Y y)D^G(Y)^{1/2}dYdy$$

$$= \Sigma_{T\in T(G)} \mid W(G,T) \mid^{-1} \int_{Z_H(F)\backslash H(F)U(F)} \int_{\mathfrak{t}^0(F)} {}^{\gamma_Y yg}\hat{f}(Y)D^G(Y)^{1/2}dYdy.$$

So we can rewrite $I_\kappa(f)$ as

$$I_\kappa(f) = \Sigma_{T\in T(G)} \mid W(G,T) \mid^{-1} \int_{\mathfrak{t}^0(F)} \int_{Z_G(F)\backslash G(F)} \hat{f}(g^{-1}Yg)\kappa(\gamma_Y^{-1}g)dgD^G(Y)^{1/2}dY.$$

For $T \in T(G), Y \in \mathfrak{t}^0(F)$, define κ_Y on $A_T(F)\backslash G(F)$ to be

$$\kappa_Y(g) = \nu(A_T) \int_{Z_G(F)\backslash A_T(F)} \kappa(\gamma_Y^{-1}ag)da. \tag{10.23}$$

Then we have

$$\begin{aligned} I_\kappa(f) =& \Sigma_{T\in T(G)}\nu(A_T)^{-1} \mid W(G,T) \mid^{-1} \\ & \times \int_{\mathfrak{t}^0(F)} \int_{A_T(F)\backslash G(F)} \hat{f}(g^{-1}Yg)\kappa_Y(g)dgD^G(Y)^{1/2}dY. \end{aligned} \tag{10.24}$$

Chapter 11

Calculation of the Limit $lim_{N\to\infty} I_{x,\omega,N}(f)$

In the last chapter, we made the transfer of the integral $I_{x,\omega,N}(f)$ to the form that is similar to the Arthur local trace formula. The only difference is that our truncated function is different from the one given by Arthur. In this chapter, we first show that we are able to change the truncated function. Then by applying Arthur's computation of the truncated function, we are going to compute the limit $\lim_{N\to\infty} I_{x,\omega,N}(f)$. This is the most technical chapter of this paper. In Section 11.1 and 11.2, we study our truncated function κ_N and introduce Arthur's truncated function. From Section 11.3 to Section 11.5, we prove that we are able to change the truncated function. In Section 11.6, we compute the limit $\lim_{N\to\infty} I_{x,\omega,N}(f)$ by applying Arthur's computation of the truncated function.

11.1 Convergence of a Premier Expression

For $x \in H_{ss}(F)$ elliptic, using the same notation as in Section 9.2, we have

$$I_{x,\omega}(f,g) = \int_{\mathfrak{h}'_x(F)} \int_{\mathfrak{h}''(F)} {}^g f^{\xi}_{x,\omega}(X' + X'')dX''dX'.$$

Then we can write $I_{x,\omega,N}(f)$ as

$$I_{x,\omega,N}(f)=\int_{\mathfrak{h}_x'(F)}\int_{H_x(F)U_x(F)\backslash G(F)}\int_{\mathfrak{h}''(F)} {}^g f_{x,\omega}^{\xi}(X'+X'')dX''\kappa_N(g)dgdX'.$$

Rewrite the two interior integrals above as

$$\int_{G_x(F)\backslash G(F)}\int_{H_x(F)U_x(F)\backslash G_x(F)}\int_{\mathfrak{h}''(F)} {}^{g''g} f_{x,\omega}^{\xi}(X'+X'')dX''\kappa_N(g''g)dg''dg.$$

After applying the formula (10.24), together with the fact that we have defined $\mathfrak{t}'=\mathfrak{h}_x'$ in Section 9.2, we have

$$\begin{aligned}I_{x,\omega,N}(f)=&\Sigma_{T\in T(G_x)}\nu(A_T\cap Z_{G_x}\backslash A_T)^{-1}\mid W(G_x,T)\mid^{-1}\\ &\times\int_{\mathfrak{t}'(F)\times(\mathfrak{t}'')^0(F)}D^{G_x}(X'')^{1/2}\\ &\times\int_{Z_{G_x}A_T(F)\backslash G(F)} {}^g f_{x,\omega}^{\sharp}(X'+X'')\kappa_{N,X''}(g)dgdX''dX'\end{aligned}\tag{11.1}$$

where

$$\kappa_{N,X''}(g)=\nu(A_T\cap Z_{G_x}\backslash A_T)\int_{Z_{G_x}\cap A_T(F)\backslash A_T(F)}\kappa_N(\gamma_{X''}^{-1}ag)da.\tag{11.2}$$

Note that the formula (10.24) is only for the case when x is in the center. However, as we explained at the beginning of Section 10, when x is not contained in the center, the computation is easier, and we can get a similar formula as (10.24) with replacing $\mathfrak{t}^0$ by $(\mathfrak{t}'')^0$ and replacing G by G_x.

Lemma 11.1.1. *For $T\in T(G_x)$, let $\omega_{T''}$ be a compact subset of $\mathfrak{t}''(F)$. There exist a rational function $Q_T(X'')$ on $\mathfrak{t}''(F)$, $k\in\mathbb{N}$ and $c>0$ such that*

$$\kappa_{N,X''}(g)\leq CN^k\sigma(g)^k(1+|\log(|Q_T(X'')|_F)|)^k(1+|\log D^{G_x}(X'')|)^k$$

for every $X''\in(\mathfrak{t}'')^0(F)\cap\omega_{T''},g\in G(F),N\geq 1$.

Proof. We first prove the following statement:

(1) There exist $c',c>0$ such that $\kappa_{N,X''}(g'g)\leq\kappa''_{c'N+c\sigma(g)}(g')$ for all $g\in G(F)$ and $g'\in G_x(F)$. Here κ''_N is the truncated function for G_x defined in the similar way as κ_N.

In fact, let $g'=m'u'k',k'g=muk$ with $m,m'\in M(F),\ u,u'\in U(F)$ and $k,k'\in K$. Then $\kappa_N(g'g)=\kappa_N(m'm)$. If this is nonzero, let

$$m'=\begin{pmatrix}m_1'&0&0\\0&m_2'&0\\0&0&m_3'\end{pmatrix},m=\begin{pmatrix}m_1&0&0\\0&m_2&0\\0&0&m_3\end{pmatrix}.$$

By the definition of κ_N (as in (5.4) and (5.5)), we have

$$\sigma((m_j')^{-1}(m_j)^{-1}m_im_i') \ll N.$$

On the other hand, we know $\sigma(m) \ll \sigma(g)$. Hence $\sigma(m_i) \ll \sigma(g)$, which implies that

$$\sigma(m_i'(m_j')^{-1}) \ll \sigma((m_j')^{-1}(m_j)^{-1}m_im_i') + \sigma(m_i) + \sigma(m_j) \ll N + \sigma(g).$$

This proves **(1)**.

Now we have

$$\begin{aligned} \kappa_{N,X''}(g) &= \nu(A_T)\int_{Z_{G_x}\cap A_T(F)\backslash A_T(F)} \kappa_N(\gamma_{X''}^{-1}ag)da \\ &\leq \nu(A_T)\int_{Z_{G_x}\cap A_T(F)\backslash A_T(F)} \kappa''_{c'N+c\sigma(g)}((\gamma_{X''})^{-1}a)da \\ &\leq \kappa''_{c'N+c\sigma(g),X''}(1). \end{aligned}$$

So it reduces to show the following:

(2) There exist an integer $k \in \mathbb{N}$, and $c > 0$ such that

$$\kappa''_{N,X''}(1) \leq cN^k(1+|\log(|Q_T(X'')|_F)|)^k(1+|\log D^{G_x}(X'')|)^k.$$

Again here we only prove for the case where x is in the center. Otherwise, we are in the lower rank case, whose proof is similar and easier. If x is in the center, $G_x = G$ and $X'' = X$. For simplicity, we will replace X'' by X, κ''_N by κ_N and $D^{G_x}(X'')$ by $D^G(X)$ for the rest of the proof. We first deal with the case when T is split. By Lemma 10.5.1, we know for $X \in \omega_T$, X_Σ belongs to a compact subset of $\Xi + \Lambda$, and $\sigma(\gamma_X) \ll 1 + |\log D^G(X)|$.

If $a \in A_T(F)$ such that $\kappa_N(\gamma_X^{-1}a) = 1$. By the definition of κ_N (as in (5.4) and (5.5)), we have $\gamma_X^{-1}a = hvy$ where $v \in U(F)$, $h \in H(F)$, and $y \in G(F)$ with $\sigma(y) \ll N$. Therefore $yXy^{-1} = v^{-1}h^{-1}X_\Sigma hv$. Since X_Σ belongs to a compact subset, $\sigma(yXy^{-1}) \ll N$, and hence

$$\sigma(v^{-1}h^{-1}X_\Sigma hv) \ll N.$$

By Lemma 10.3.1, the isomorphism (10.2) is algebraic. This implies $\sigma(v) \ll N$ and $\sigma(h^{-1}X_\Sigma h) \ll N$.

Now let

$$X_\Sigma = \begin{pmatrix} 0 & 0 & Z \\ aI_2 & 0 & Y \\ 0 & bI_2 & 0 \end{pmatrix}.$$

By Proposition 2.4.2, we can find $s \in \mathrm{GL}_2(E)$ such that $s^{-1}Zs$ is a diagonal matrix and $\sigma(s) \ll 1 + |\log(D^{GL_2(E)}(s^{-1}Zs))|$. Here E/F is a finite extension generated by the elements in $F^\times/(F^\times)^2$. Note that $D^{\mathrm{GL}_2(E)}(s^{-1}Zs) = \mathrm{tr}(Z)^2 - 4\det(Z)$, while the right hand side can be expressed as a polynomial of the coefficients of the characteristic polynomial of X_Σ, so it can be expressed as a polynomial on $\mathfrak{t}_0(F)$. We remark that if x is not in center, this will be polynomial on $\mathfrak{t}''(F)$.

After conjugating by s, we may assume that Z is a diagonal matrix with distinct eigenvalues λ_1 and λ_2 (we only need to change h to sh). Here the eigenvalues are distinct because of the "generic position" assumption. After multiplying by elements in the center and in the open compact subgroup, together with the Iwasawa decomposition, we may assume that

$$h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} Y \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Since $\sigma(h^{-1}X_\Sigma h) \ll N$, we have $\sigma(h^{-1}Zh), \sigma(h^{-1}Yh) \ll N$. This implies

$$\sigma(x(\lambda_1 - \lambda_2)), \sigma(Ay_{12}), \sigma(A^{-1}y_{21}) \ll N.$$

Here for $t \in F$, $\sigma(t) = \log(\max\{1, |t|\})$. Therefore, we obtain that $\sigma(x) \ll \max\{1, N - \log(|\lambda_1 - \lambda_2|)\}$. Here Z and Y belong to a fixed compact subset before conjugation. Furthermore, after conjugating by s and $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\sigma(Y) \ll \sigma(s) + \sigma(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})$. So we have

$$\begin{aligned} \sigma(A) &\ll \max\{1, N - \sigma(y_{12})\} \\ &\ll \max\{1, N + \sigma(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) + \sigma(s) - \sigma(y_{12}y_{21})\} \end{aligned} \tag{11.3}$$

and

$$\begin{aligned}\sigma(A^{-1}) &\ll \max\{1, N-\sigma(y_{21})\} \\ &\ll \max\{1, N+\sigma(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix})+\sigma(s)-\sigma(y_{12}y_{21})\}. \end{aligned} \tag{11.4}$$

Note that here by the "generic position" assumption, we have $y_{12}y_{21}\neq 0$.

Recall that as in the proof of Lemma 10.4.1, we have the following relations between the coefficients of the characteristic polynomial of X_Σ and the data given by Z and Y:

$$\text{coefficient of } \lambda^4 = b\operatorname{tr}(Y) := ba_4,$$

$$\text{coefficient of } \lambda^3 = ab\operatorname{tr}(Z) := aba_3,$$

$$\text{coefficient of } \lambda^2 = b^2\det(Y) := b^2a_2,$$

$$\text{coefficient of } \lambda = ab^2(\lambda \text{ coefficient of } \det(Z+\lambda Y)) := ab^2a_1,$$

and

$$\text{coefficient of } \lambda^0 = a^2b^2\det(Z) := a^2b^2a_0.$$

Then

$$\begin{cases} y_{11}+y_{22}=a_4 \\ \lambda_1y_{11}+\lambda_2y_{22}=a_1 \end{cases}$$

and

$$\begin{cases} \lambda_1+\lambda_2=a_3 \\ \lambda_1\lambda_2=a_0 \end{cases}.$$

This implies

$$\begin{cases} y_{11}=\frac{a_1-\lambda_1a_4}{\lambda_2-\lambda_1} \\ y_{22}=\frac{\lambda_2a_4-a_1}{\lambda_2-\lambda_1} \end{cases}.$$

So we have

$$y_{11}y_{22}=-\frac{\lambda_1\lambda_2a_4^2-a_1a_4(\lambda_1+\lambda_2)+a_1^2}{(\lambda_1-\lambda_2)^2}=\frac{a_0a_4^2-a_1a_3a_4+a_1^2}{a_3^2-4a_0}.$$

In particular, $y_{12}y_{21}=\det(Y)-y_{11}y_{22}=a_2-y_{11}y_{22}$ is a rational function of the a_i's, and hence it is a rational function on $\mathfrak{t}_0(F)$. Also

$$\sigma(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix})=\sigma(x)\ll\max\{1, N-\log(|\lambda_1-\lambda_2|)\} \tag{11.5}$$

where the right hand side can be expressed as a logarithmic function of some rational function on $\mathfrak{t}_0(F)$.

Finally, combining (11.3), (11.4), (11.5), and the majorization of s, we can find a rational function $Q_T(X)$ on $\mathfrak{t}_0(F)$ such that $\sigma(h) \ll N + (1 + \log|Q_T(X)|)$. Then combining the majorization of v, y and γ_Y, we know that up to an element in the center, if $\kappa_N(\gamma_X^{-1}a) = 1$, we have

$$\sigma(a) \ll N + (1 + \log Q_T(X)) + (1 + \log D^G(X)). \tag{11.6}$$

Since $mes\{a \in (Z_{G_x} \cap A_T(F))\backslash A_T(F) \mid \sigma_{Z_{G_x}\backslash G_x}(a) \leq r\} \ll r^k$ for some $k \in \mathbb{N}$, the Lemma follows from the definition of $\kappa_{N,X''}$ (as in (11.2)).

Now if T is not split, since we are talking about majorization, we may pass to a finite extension. Then by the same argument as above, we can show that if $\kappa_N(\gamma_X^{-1}a) = 1$ for some $a \in A_T(F)$, up to an element in the center, the estimation (11.6) will still holds. Then we can still prove the lemma as in the split case. □

Now let $\mathcal{Q}_T$ be a finite set of polynomials on $\mathfrak{t}''(F)$ that contains $D^{G_x}(X'')$. The denominator and numerator of $Q_T(X'')$ and some other polynomials that will be defined later in Section 11.5. For $l > 0$, let $\mathfrak{t}_0(F)[\leq l]$ be the set of $X = X' + X'' \in \mathfrak{t}_0(F)$ such that there exists $Q \in \mathcal{Q}_T$ with $|Q(X'')|_F \leq l$, and let $\mathfrak{t}_0(F)[> l]$ be its complement in $\mathfrak{t}_0(F)$. We define $I_{N,\leq l}$ to be the integral of the expression of $I_{x,\omega,N}(f)$ restricted on $(\mathfrak{t}'(F) \times (\mathfrak{t}'')^0(F)) \cap \mathfrak{t}_0(F)[\leq l]$ (as in (11.1)). Similarly we can define $I_{N,>l}$. We then have

$$I_{x,\omega,N}(f) = I_{N,\leq l} + I_{N,>l}. \tag{11.7}$$

Lemma 11.1.2. *The following statements hold.*

1. *There exist $k \in \mathbb{N}$ and $c > 0$ such that $\mid I_{x,\omega,N}(f) \mid \leq cN^k$ for all $N \geq 1$.*

2. *There exist $b \geq 1$ and $c > 0$ such that $\mid I_{N,\leq N^{-b}} \mid \leq cN^{-1}$ for all $N \geq 1$.*

Proof. By condition (5) of a good neighborhood (as in Definition 3.1.1), there exists a compact subset $\Gamma \subset G(F)$ such that $({}^g f_{x,\omega}^{\xi})\hat{} = 0$ if $g \notin G_x(F)\Gamma$.

By replacing $Z_{G_x}A_T(F)\backslash G(F)$ by $Z_{G_x}A_T(F)\backslash G_x \cdot \gamma$ for some $\gamma \in \Gamma$, we can majorize ${}^{\gamma} f_{x,\omega}^{\sharp}$ by a linear combination of functions $f' \otimes f''$ where $f' \in C_c^\infty(\mathfrak{g}_x'(F))$, and $f'' \in$

$C_c^\infty(\mathfrak{g}''(F))$. So the integral in (11.1) is majored by

$$\int_{\mathfrak{t}'(F)\times(\mathfrak{t}'')^0(F)} D^{G_x}(X'')^{1/2}\int_{Z_{G_x}A_T(F)\backslash G_x(F)} f'(X')f''(g^{-1}X''g)\kappa_{N,X''}(\gamma g)dgdX''dX'. \tag{11.8}$$

Now we fix a compact subset $\omega_{T''}\subset\mathfrak{t}''(F)$ such that for every $g\in G_x(F)$, the function $X''\to f''(g^{-1}X''g)$ on $\mathfrak{t}''(F)$ is supported on $\omega_{T''}$. By Proposition 2.4.2, up to an element in $Z_{G_x}(F)A_T(F)$, we may choose g such that $\sigma(g)\ll 1+|\log(D^{G_x}(X''))|$. Using the lemma above, we have

$$\kappa_{N,X''}(\gamma g)\ll N^k\phi(X'')$$

where

$$\phi(X'')=(1+|\log(|Q_T(X'')|_F)|)^k(1+|\log(D^{G_x}(X''))|)^{2k}.$$

So the expression (11.8) is majored by

$$N^k\int_{\mathfrak{t}'(F)\times(\mathfrak{t}'')^0(F)} D^{G_x}(X'')^{1/2}\int_{Z_{G_x}A_T(F)\backslash G_x(F)} f'(X')f''(g^{-1}X''g)\phi(X'')dgdX''dX.$$

This is majored by

$$N^k\int_{\mathfrak{t}_0(F)} J_{G_x}(X'+X'',f'\otimes f'')\phi(X'')dX''dX' \tag{11.9}$$

where J_{G_x} is the orbital integral. Due to the work of Harish-Chandra, the orbital integral is always bounded, and hence (11.9) is majored by

$$N^k\int_\omega \phi(X'')dX''dX' \tag{11.10}$$

where ω is a compact subset of $\mathfrak{t}_0(F)$. By Lemma 2.4 of [W10], $\phi(X)$ is locally integrable, and hence the integral in (11.10) is convergent. This finishes the proof of the first part.

For the second part, by the same argument, we have majorization

$$|\, I\,|_{N,\leq N^{-b}}\ll N^k\int_{\omega\cap\mathfrak{t}_0(F)[\leq N^{-b}]}\phi(X)dX.$$

Then, by the Schwartz inequality, the right hand side is majored by

$$\begin{aligned} & N^k(\int_{\omega\cap\mathfrak{t}_0(F)[\leq N^{-b}]} dX)^{1/2}(\int_{\omega\cap\mathfrak{t}_0(F)[\leq N^{-b}]}\phi(X)^2dX)^{1/2} \\ \ll\ & N^k\cdot\Sigma_{Q\in\mathcal{Q}_T} mes\{X\in\omega\mid |\,Q(X)\,|_F\leq N^{-b}\}\ll N^k(N^{-b})^r \end{aligned}$$

for some $r>0$ that only depends on the dimension of $\mathfrak{t}_0$. Now we just need to let b large such that $N^k(N^{-b})^r\ll N^{-1}$. This finishes the proof of the Lemma. $\square$

Definition 11.1.3. *With the notations above, let* $I^*_{x,\omega,N}(f) = I_{N,>N^{-b}}$.

By the Lemma above, we have

$$\lim_{N\to\infty}(I_{x,\omega,N}(f) - I^*_{x,\omega,N}(f)) = 0. \tag{11.11}$$

11.2 Combinatorial Definition

Fix $T \in T(G_x)$, let $M_\sharp$ be the centralizer of A_T in G. This is a Levi subgroup of G, it is easy to check that $A_T = A_{M_\sharp}$. Since x is elliptic, we know $Z_{G_x} \cap A_T = Z_G$ for any $T \in T(G_x)$, and hence we have $\nu(A_T \cap Z_{G_x} \backslash A_T) = \nu(Z_G \backslash A_T) = \nu(A_T)$. Note that we always choose the Haar measure on G so that $\nu(Z_G) = 1$.

Let $\mathcal{Y} = (Y_{P_\sharp})_{P_\sharp \in \mathcal{P}(M_\sharp)}$ be a family of elements in $\mathfrak{a}_{M_\sharp}$ that are $(G, M_\sharp)$-orthogonal and positive. Then for $Q = LU_Q \in \mathcal{F}(M_\sharp)$, let $\zeta \to \sigma^Q_{M_\sharp}(\zeta, \mathcal{Y})$ be the characteristic function on $\mathfrak{a}_{M_\sharp}$ that supports on the sum of $\mathfrak{a}_L$ and the convex envelop generated by the family $(Y_{P_\sharp})_{P_\sharp \in \mathcal{P}(M_\sharp), P_\sharp \subset Q}$. Let τ_Q be the characteristic function on $\mathfrak{a}_{M_\sharp}$ that supports on $\mathfrak{a}^L_{M_\sharp} + \mathfrak{a}^+_Q$. The following proposition follows from 3.9 of [Ar91].

Proposition 11.2.1. *The function*

$$\zeta \to \sigma^Q_{M_\sharp}(\zeta, \mathcal{Y})\tau_Q(\zeta - Y_Q)$$

is the characteristic function on $\mathfrak{a}_{M_\sharp}$*, whose support is on the sum of* $\mathfrak{a}^+_Q$ *and the convex envelope generated by* $(Y_{P_\sharp})_{P_\sharp \in \mathcal{P}(M_\sharp), P_\sharp \subset Q}$*. Moreover, for every* $\zeta \in \mathfrak{a}_{M_\sharp}$*, the following identity holds.*

$$\Sigma_{Q \in \mathcal{F}(M_\sharp)} \sigma^Q_{M_\sharp}(\zeta, \mathcal{Y})\tau_Q(\zeta - Y_Q) = 1. \tag{11.12}$$

11.3 Change the Truncated Function

We use the same notation as Section 11.2. Fix a minimal Levi subgroup M_{min} of G contained in $M_\sharp$, a maximal open compact subgroup K_{min} of G in good position with respect to M_{min} and $P_{min} = M_{min}U_{min} \in \mathcal{P}(M_{min})$. Let Δ_{min} be the set of simple roots of $A_{M_{min}}$ in $\mathfrak{u}_{min}$. Given $Y_{min} \in \mathfrak{a}^+_{P_{min}}$, for any $P' \in \mathcal{P}(M_{min})$, there exists a unique element $w \in W(G, M_{min})$ such that $wP_{min}w^{-1} = P'$. Set $Y_{P'} = wY_{P_{min}}$. The family $(Y_{P'})_{P'\in\mathcal{P}(M_{min})}$ is (G, M_{min})-orthogonal and positive. For $g \in G(F)$, define $\mathcal{Y}(g) = (Y(g)_Q)_{Q\in\mathcal{P}(M_\sharp)}$ to be

$$Y(g)_Q = Y_Q - H_{\bar{Q}}(g).$$

Then it is easy to show the following statements.

(1) There exists $c_1 > 0$ such that for any $g \in G(F)$ with $\sigma(g) < c_1 \inf\{\alpha(Y_{P_{min}}); \alpha \in \Delta_{min}\}$, the family $\mathcal{Y}(g)$ is $(G, M_\sharp)$-orthogonal and positive. And $Y(g)_Q \in \mathfrak{a}_Q^+$ for all $Q \in \mathcal{F}(M_\sharp)$.

We fix such a c_1. Note that for $m \in M_\sharp(F)$, $\mathcal{Y}(mg)$ is a translation of $\mathcal{Y}(g)$ by $H_{M_\sharp}(m)$. Hence $\mathcal{Y}(g)$ is $(G, M_\sharp)$-orthogonal and positive for

$$g \in M_\sharp(F)\{g' \in G(F) \mid \sigma(g') < c_1 \inf\{\alpha(Y_{P_{min}}); \alpha \in \Delta_{min}\}\}.$$

For such g, let

$$\tilde{v}(g) = \nu(A_T) \int_{Z_G(F)\backslash A_T(F)} \sigma_{M_\sharp}^G(H_{M_\sharp}(a), \mathcal{Y}(g))da. \tag{11.13}$$

(2) There exist $c_2 > 0$ and a compact subset ω_T of $\mathfrak{t}_0(F)$ satisfying the following condition: If $g \in G(F)$, and

$$X \in \mathfrak{t}_0(F)[> N^{-b}] \cap (\mathfrak{t}'(F) \times (\mathfrak{t}'')^0(F))$$

with $({}^g f_{x,\omega})^\sharp(X) \neq 0$, then $X \in \omega_T$ and $\sigma_T(g) < c_2 \log(N)$.

In fact, since $({}^g f_{x,\omega})^\sharp(X) = (f_{x,\omega})\hat{}(g^{-1}Xg)$, $g^{-1}Xg$ is contained in compact subset of $\mathfrak{g}_{x,0}(F)$. This implies that X belongs to a compact subset of $\mathfrak{t}_0(F)$. By Proposition 2.4.2, we have

$$\sigma_T(g) \ll 1 + \mid \log D^{G_x}(X) \mid = 1 + \mid \log D^{G_x}(X'') \mid \ll \log(N)$$

where the last inequality holds because $X \in \mathfrak{t}_0(F)[> N^{-b}]$ and belongs to a compact subset.

Now we fix ω_T and c_2 as in (2). We may assume that $\omega_T = \omega_{T'} \times \omega_{T''}$ where $\omega_{T'}$ is a compact subset of $\mathfrak{t}'(F)$ and $\omega_{T''}$ is a compact subset of $\mathfrak{t}''(F)$. Suppose that

$$c_2 \log(N) < c_1 \inf\{\alpha(Y_{min}) \mid \alpha \in \Delta_{min}\}.$$

Here c_1 comes from (1). Then $\tilde{v}(g)$ is defined for all $g \in G(F)$ satisfying condition (2).

Proposition 11.3.1. *There exist $c > 0$ and $N_0 \geq 1$ such that if $N \geq N_0$ and $c\log(N) < \inf\{\alpha(Y_{min}) \mid \alpha \in \Delta_{min}\}$, we have*

$$\int_{Z_{G_x}(F)A_T(F)\backslash G(F)} {}^g f^{\sharp}_{x,\omega}(X)\kappa_{N,X''}(g)dg = \int_{Z_{G_x}(F)A_T(F)\backslash G(F)} {}^g f^{\sharp}_{x,\omega}(X)\tilde{v}(g)dg \quad (11.14)$$

for every $X \in \mathfrak{t}_0(F)[> N^{-b}] \cap (\mathfrak{t}'(F) \times (\mathfrak{t}'')^0(F))$.

Proof. For any $Z_{P_{min}} \in \mathfrak{a}^+_{P_{min}}$, replacing $Y_{P_{min}}$ by $Z_{P_{min}}$, we can construct the family $\mathcal{Z}(g)$ in the same way as $\mathcal{Y}(g)$. Assume that

$$c_2 \log(N) < c_1 \inf\{\alpha(Z_{min}) \mid \alpha \in \Delta_{min}\}. \quad (11.15)$$

For $g \in G(F)$ with $\sigma(g) < c_2 \log(N)$, $\mathcal{Z}(g)$ is still $(G, M_\sharp)$-orthogonal and positive. So for $a \in A_T(F)$, by Proposition 11.2.1, we have

$$\Sigma_{Q\in F(M_\sharp)}\sigma^Q_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q) = 1.$$

Then we know

$$\tilde{v}(g) = \nu(A_T)\Sigma_{Q\in F(M_\sharp)}\tilde{v}(Q, g) \quad (11.16)$$

and

$$\kappa_{N,X''}(g) = \nu(A_T)\Sigma_{Q\in F(M_\sharp)}\kappa_{N,X''}(Q, g) \quad (11.17)$$

where

$$\tilde{v}(Q, g) = \int_{Z_G\backslash A_T(F)} \sigma^G_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Y}(g))\sigma^Q_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q)da \quad (11.18)$$

and

$$\kappa_{N,X''}(Q, g) = \int_{Z_G\backslash A_T(F)} \kappa_N(\gamma^{-1}_{X''}ag)\sigma^Q_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q)da. \quad (11.19)$$

(3) The functions $g \to \tilde{v}(Q, g)$ and $g \to \kappa_{N,X''}(Q, g)$ are left$A_T(F)$-invariant.

Since for $t \in A_T(F)$, $H_{P'}(tg) = H_{M_\sharp}(t) + H_{P'}(g)$ for all $P' \in \mathcal{P}(M_\sharp)$. We can just change variable $a \to at$ in the definition of $\tilde{v}(Q, g)$ and $\kappa_{N,X''}(Q, g)$. This gives us the left $A_T(F)$-invariant of both functions, and proves (3).

Now for $X \in \mathfrak{t}'(F) \times (\mathfrak{t}'')^0(F)$, we have

$$\int_{Z_{G_x}A_T(F)\backslash G(F)} {}^g f^{\sharp}_{x,\omega}(X)\kappa_{N,X''}(g)dg = \nu(A_T)\Sigma_{Q\in\mathcal{F}(M_\sharp)}I(Q, X) \quad (11.20)$$

and

$$\int_{Z_{G_x}A_T(F)\backslash G(F)} {}^g f^{\sharp}_{x,\omega}(X)\tilde{v}(g)dg = \nu(A_T)\Sigma_{Q\in\mathcal{F}(M_\sharp)}J(Q,X) \tag{11.21}$$

where

$$I(Q,X) = \int_{Z_{G_x}A_T(F)\backslash G(F)} {}^g f^{\sharp}_{x,\omega}(X)\kappa_{N,X''}(Q,g)dg \tag{11.22}$$

and

$$J(Q,X) = \int_{Z_{G_x}A_T(F)\backslash G(F)} {}^g f^{\sharp}_{x,\omega}(X)\tilde{v}(Q,g)dg. \tag{11.23}$$

Then it is enough to show that for all $Q \in \mathcal{F}(M_\sharp)$, $I(Q,X) = J(Q,X)$.

Firstly we consider the case when $Q = G$. Suppose

$$\sup\{\alpha(Z_{P_{min}}) \mid \alpha \in \Delta_{min}\} \leq \begin{cases} \inf\{\alpha(Y_{P_{min}}) \mid \alpha \in \Delta_{min}\}, \\ \log(N)^2. \end{cases} \tag{11.24}$$

Then we are going to prove

(4) There exists $N_1 > 1$ such that for all $N \geq N_1$, $g \in G(F)$ with $\sigma_T(g) \leq c_2 \log(N)$, and for all $X'' \in \omega_{T''} \cap (\mathfrak{t}'')^0(F)[> N^{-b}]$, we have

$$\kappa_{N,X''}(G,g) = \tilde{v}(G,g). \tag{11.25}$$

Here $\mathfrak{t}''[> N^{-b}]$ means that we only consider the polynomials $D^{G_x}(X'')$ together with the numerator and the denominator of $Q_T(X'')$ which are elements in $\mathcal{Q}_T$.

In order to prove (4), it is enough to show that for all $a \in A_T(F)$ with $\sigma^G_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Z}(g)) = 1$, we have $\sigma^G_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Y}(g)) = \kappa_N(\gamma_X^{-1}ag)$. Since both sides of (11.25) are left $A_T(F)$-invariant, we may assume that $\sigma(g) \leq c_2 \log(N)$.

By the first inequality of (11.24), $\sigma^G_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Z}(g)) = 1$ will implies

$$\sigma^G_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Y}(g)) = 1.$$

Then by the second inequality of (11.24), together with the fact that $\sigma(g) \ll \log(N)$, we know $\mid \mathcal{Z}(g)_{P'} \mid \ll \log(N)^2$ for every $P' \in \mathcal{P}(M_\sharp)$, here $|\cdot|$ is the norm on $\mathfrak{a}_{M_\sharp}/\mathfrak{a}_G$. Then combining with the fact that $\sigma^G_{M_\sharp}(H_{M_\sharp}(a), \mathcal{Z}(g)) = 1$, we know that up to an element in the center, $\sigma(a) \ll \log(N)^2$. Since the integrals defining $I(Q,X)$ and $J(Q,X)$ are integrating modulo the center, we may just assume that $\sigma(a) \ll \log(N)^2$.

By (10.22) and the fact that $X'' \in \omega_{T''} \cap (\mathfrak{t}'')^0(F)[> N^{-b}]$, we know $\sigma(\gamma_X) \ll 1 + \mid \log D^{G_x}(X) \mid \ll \log(N)$, and hence $\sigma(\gamma_X^{-1}ag) \ll \log(N)^2$. By the definition of κ_N, together with the relations between the norm of an element and the norm of its Iwasawa decomposition, we can find $c_3 > 0$ such that for any $g' \in G(F)$ with $\sigma(g') < c_3 N$, we have $\kappa_N(g') = 1$. Now for N large enough, we definitely have $\sigma(\gamma_X^{-1} a g) < c_3 N$. In this case, we have $\kappa_N(\gamma_X^{-1} a g) = 1 = \sigma_{M_\sharp}^G(H_{M_\sharp}(a), \mathcal{Y}(g))$. This proves (4).

Combining (2) and (4), together with (11.22) and (11.23), we have

$$I(G, X) = J(G, X) \tag{11.26}$$

for every $N \geq N_1, X \in \mathfrak{t}_0(F)[> N^{-b}] \cap (\mathfrak{t}'(F) \times (\mathfrak{t}'')^0(F))$.

Now for $Q = LU_Q \in \mathcal{F}(M_\sharp)$ **with** $Q \neq G$ We can decompose the integrals in (11.22) and (11.23) by

$$I(Q, X) = \int_{K_{min}} \int_{Z_{G_x} A_T(F) \backslash L(F)} \int_{U_{\bar{Q}}(F)} {}^{\bar{u}lk} f_{x,\omega}^\sharp(X) \kappa_{N,X''}(Q, \bar{u}lk) d\bar{u} \delta_Q(l) dl dk \tag{11.27}$$

and

$$J(Q, X) = \int_{K_{min}} \int_{Z_{G_x} A_T(F) \backslash L(F)} \int_{U_{\bar{Q}}(F)} {}^{\bar{u}lk} f_{x,\omega}^\sharp(X) \tilde{v}(Q, \bar{u}lk) d\bar{u} \delta_Q(l) dl dk. \tag{11.28}$$

The following two properties will be proved in Section 11.4 and 11.5.

(5) If $g \in G(F)$ and $\bar{u} \in U_{\bar{Q}}(F)$ with

$$\sigma(g), \sigma(\bar{u}g) < c_1 \inf\{\alpha(Z_{P_{min}}) \mid \alpha \in \Delta_{min}\},$$

then $\tilde{v}(Q, \bar{u}g) = \tilde{v}(Q, g)$.

(6) Given $c_4 > 0$, we can find $c_5 > 0$ such that if

$$c_5 \log(N) < \inf\{\alpha(Z_{P_{min}}) \mid \alpha \in \Delta_{min}\},$$

we have $\kappa_{N,X''}(Q, \bar{u}g) = \kappa_{N,X''}(Q, g)$ for all $g \in G(F)$ and $\bar{u} \in U_{\bar{Q}}(F)$ with $\sigma(g), \sigma(\bar{u}), \sigma(\bar{u}g) < c_4 \log(N)$, and for all $X'' \in \omega_{T''} \cap (\mathfrak{t}'')^0(F))[> N^{-b}]$.

Based on (5) and (6), we are going to show:

(7) There exists $c_5 > 0$ such that if

$$c_5 \log(N) < \inf\{\alpha(Z_{P_{min}}) \mid \alpha \in \Delta_{min}\}, \tag{11.29}$$

we have $I(Q, X) = J(Q, X) = 0$ for all $X \in \mathfrak{t}_0(F)[> N^{-b}] \cap (\mathfrak{t}'(F) \times (\mathfrak{t}'')^0(F))$.

In fact, by (2), we may assume that $X \in \omega_T$. We first consider $I(Q,X)$. By (2), we can restrict the integral $\int_{Z_{G_x}A_T(F)\backslash L(F)}$ in (11.27) to those l for which there exist $\bar{u} \in U_{\bar{Q}}(F)$ and $K \in K_{min}$ such that $\sigma_T(\bar{u}lk) < c_2\log(N)$. Then up to an element in $A_T(F)$, l can be represented by an element in $L(F)$ such that $\sigma(l) < c_6\log(N)$ for some constant c_6. We can find $c_7 > 0$ such that for all $l, \bar{u}$ and k with $\sigma(l) < c_6\log(N)$ and $\sigma(\bar{u}lk) < c_2\log(N)$, we have $\sigma(\bar{u}) < c_7\log(N)$. Now let $c_4 = c_2 + c_7$, and choose c_5 as in (6). Then by applying (6), we know that for fixed $k \in K_{min}, l \in L(F)$ with $\sigma(l) < c_6\log(N)$, we have

$$ {}^{\bar{u}lk}f^{\sharp}_{x,\omega}(X)\kappa_{N,X''}(Q,\bar{u}lk) = {}^{\bar{u}lk}f^{\sharp}_{x,\omega}(X)\kappa_{N,X''}(Q,lk) \tag{11.30} $$

for all $\bar{u} \in U_{\bar{Q}}(F)$. On the other hand, if $\sigma(\bar{u}lk) \geq c_2\log(N)$, both side of (11.30) are equal to 0 by (2). Therefore (11.30) holds for all $\bar{u}$, l and k.

From (11.30), we know that in the expression of $I(Q,X)$ (as in (11.27)), the inner integral is just

$$ \int_{U_{\bar{Q}}(F)} {}^{\bar{u}lk}f^{\sharp}_{x,\omega}(X)d\bar{u}. $$

This is zero for $Q \neq G$ by Lemma 3.7.2. Hence $I(Q,X) = 0$. By applying the same argument except replacing (6) by (5), we can also show that $J(Q,X) = 0$. This proves (7), and finishes the proof of the Proposition.

The last thing we need to do is to verify that we can find $Z_{P_{min}}$ satisfies condition (11.15), (11.24) and (11.29). This just follows from the conditions we imposed on N and $Y_{P_{min}}$. $\square$

11.4 Proof of 11.3(5)

By (11.18), we have

$$ \tilde{v}(Q,G) = \int_{Z_G(F)\backslash A_T(F)} \sigma^G_{M_\sharp}(H_{M_\sharp}(a),\mathcal{Y}(g))\sigma^Q_{M_\sharp}(H_{M_\sharp}(a),\mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a)-\mathcal{Z}(g)_Q)da. $$

The function $\zeta \to \sigma^Q_{M_\sharp}(\zeta,\mathcal{Z}(g))$ and $\zeta \to \tau_Q(\zeta - \mathcal{Z}(g)_Q)$ only depend on $H_{\bar{P}'}(g)$ for $P' \in \mathcal{F}(M_\sharp)$ with $P' \subset Q$. For such P', $H_{\bar{P}'}(\bar{u}g) = H_{\bar{P}'}(g)$ for $\bar{u} \in U_{\bar{Q}}(F)$. Therefore for all $\bar{u} \in U_{\bar{Q}}(F)$, we have

$$ \sigma^Q_{M_\sharp}(H_{M_\sharp}(a),\mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a)-\mathcal{Z}(g)_Q) = \sigma^Q_{M_\sharp}(H_{M_\sharp}(a),\mathcal{Z}(\bar{u}g))\tau_Q(H_{M_\sharp}(a)-\mathcal{Z}(\bar{u}g)_Q). $$

Now for all $a \in A_T(F)$ with the property that

$$\sigma_{M_\sharp}^Q(H_{M_\sharp}(a), \mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q) \neq 0,$$

we need to show

$$\sigma_{M_\sharp}^G(H_{M_\sharp}(a), \mathcal{Y}(g)) = \sigma_{M_\sharp}^G(H_{M_\sharp}(a), \mathcal{Y}(\bar{u}g)). \tag{11.31}$$

For any $P' \in \mathcal{P}(M_\sharp)$ with $P' \subset Q$, it determines a chamber $\mathfrak{a}_{P'}^{L,+}$ in $\mathfrak{a}_{M_\sharp}^L$. Let $\zeta = H_{M_\sharp}(a)$, and fix a P' such that $proj_{M_\sharp}^L(\zeta) \in CL(\mathfrak{a}_{P'}^{L,+})$ where CL means closure.

Lemma 11.4.1. *$\zeta \in CL(\mathfrak{a}_{P'}^+)$.*

Proof. By the definition of the functions $\sigma_{M_\sharp}^Q$ and τ_Q, together with the fact that $\sigma_{M_\sharp}^Q(H_{M_\sharp}(a), \mathcal{Y}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q) \neq 0$, we know that ζ is the summation of an element $\zeta' \in \mathfrak{a}_Q^+$ and an element ζ'' belonging to the convex envelop generated by $\mathcal{Z}(g)_{P''}$ for $P'' \in \mathcal{P}(M_\sharp)$ with $P'' \subset Q$. For any root α of $A_{M_\sharp}$ in $\mathfrak{g}$, positive with respect to P', if α is in U_Q, then it is positive for all $P'' \subset Q$ above. By 11.3(1), $\mathcal{Z}(g)_{P''} \in \mathfrak{a}_{P''}^+$, and $\alpha(\zeta'') > 0$. Also we know $\alpha(\zeta') > 0$ because α is in U_Q and $\zeta' \in \mathfrak{a}_Q^+$. Combining these two inequalities, we have $\alpha(\zeta) > 0$.

If α is in $U_{P'} \cap L$, then $\alpha(\zeta) = \alpha(proj_{M_\sharp}^L(\zeta)) \geq 0$ by the choice of P'. So the lemma follows. $\square$

By Lemma 3.1 of [Ar91], for $\zeta \in CL(\mathfrak{a}_{P'}^+)$, $\sigma_{M_\sharp}^G(\zeta, \mathcal{Y}(g)) = 1$ is equivalent to certain inequality on $\zeta - \mathcal{Y}(g)_{P'}$. This only depends on $H_{\bar{P}'}(g)$. Since $P' \subset Q$ and $H_{\bar{P}'}(g) = H_{\bar{P}'}(\bar{u}g)$, (11.31) follows. This proves 11.3(5).

11.5 Proof of 11.3(6)

Same as in Section 10.6, we fix a map $X'' \to \gamma_{X''}$ such that

1. There exists a compact subset Ω of $\Xi + \Sigma$ such that $X''_\Sigma = \gamma_{X''}^{-1} X'' \gamma_{X''} \in \Omega$ for all $X'' \in \omega_{T''} \cap (\mathfrak{t}'')^0(F)$.

2. There exists $c_1 > 0$ such that $\sigma(\gamma_{X''}) < c_1 \log(N)$ for all $X'' \in \omega_{T''} \cap (\mathfrak{t}'')^0(F)[> N^{-b}]$

For $Q = LU_Q \in \mathcal{F}(M_\sharp)$, let Σ_Q^+ be the roots of $A_{M_\sharp}$ in $\mathfrak{u}_Q$.

Lemma 11.5.1. *For $c > 0$, there exists $c' > 0$ satisfying the following condition: For given $a \in A_T(F)$, $g \in G(F)$, $\bar{u} \in U_{\bar{Q}}(F)$ and $X'' \in \omega_{T''} \cap (\mathfrak{t}'')^0(F)[> N^{-b}]$, assume that $\sigma(g), \sigma(\bar{u}), \sigma(\bar{u}g) < c\log(N)$, and $\alpha(H_{M_\sharp}(a)) > c'\log(N)$ for all $\alpha \in \Sigma_Q^+$. Then $\kappa_N(\gamma_{X''}^{-1} a\bar{u}g) = \kappa_N(\gamma_{X''}^{-1} ag)$.*

Proof. We first prove:

(3) It's enough to treat the case when $T \in T(G_x)$ is split.

In fact, if F'/F is a finite extension, we can still define $\kappa_N^{F'}$ on $G(F')$ in the same way as κ_N. It is easy to see that $\kappa_N^{F'} = \kappa_{Nval_{F'}(\varpi_F)}$ on $G(F)$, and hence we can pass to a finite extension of F. Therefore we may assume that T and G_x are split. This proves (3).

(4) Let $X'' \to \underline{\gamma_{X''}}, \underline{X_\Sigma}'' = (\underline{\gamma_{X''}})^{-1} X \underline{\gamma_{X''}}$ be another local sections satisfying Conditions (1) and (2). Then the lemma holds for $\gamma_{X''}, X''_\Sigma$ if and only if it holds for $\underline{\gamma_{X''}}, \underline{X_\Sigma}''$.

For $X'' \in \mathfrak{t}''(F)$, by Lemma 10.4.1, there exist $u(X'') \in U_x(F)$ and $t(X'') \in H_x(F)$ such that

$$\underline{X_\Sigma}'' = u(X'')^{-1} t(X'')^{-1} X''_\Sigma t(X'') u(X'').$$

By the choice of X''_Σ, we have $t(X'')^{-1} X''_\Sigma t(X'') \in \Xi + \Lambda$. It follows that $u(X'')$ and $t(X'')^{-1} X''_\Sigma t(X'')$ can be expressed in terms of polynomials of $\underline{X_\Sigma}''$. Hence they are bounded. By Lemma 11.1.1, we know

$$\sigma(t(X'')) \ll 1 + \mid \log \mid Q_T(X'') \mid_F \mid .$$

So for $X'' \in (\mathfrak{t}'')^0(F)[> N^{-b}] \cap \omega_{T''}$, we have $\sigma(t(X'')) \ll \log(N)$.

Note that the conjugations of X'' by $\underline{\gamma_{X''}}$ and by $\gamma_{X''} t(X'') u(X'')$ are the same. Since X'' is regular, there exists $y(X'') \in T(F)$ such that $\underline{\gamma_{X''}} = y(X'') \gamma_{X''} t(X'') u(X'')$. The majorization of $\underline{\gamma_{X''}}$, $\gamma_{X''}, t(X'')$, and $u(X'')$ implies that $\sigma(y(X'')) \ll \log(N)$ for $X \in \mathfrak{t}_0(F)[> N^{-b}] \cap \omega_T$. Let $c > 0, a, g, \bar{u}, X''$ be as in the statement of lemma. Since κ_N is left $H(F)U(F)$-invariant, we have

$$\kappa_N((\underline{\gamma_{X''}})^{-1} a\bar{u}g) = \kappa_N(\gamma_{X''}^{-1} a\bar{u}'g'), \kappa_N((\underline{\gamma_{X''}})^{-1} ag) = \kappa_N(\gamma_{X''}^{-1} ag')$$

where $g' = y(X'')^{-1} g$ and $\bar{u}' = y(X'')^{-1} \bar{u} y(X'')$.

Now suppose that the Lemma holds for $\gamma_{X''}, X''_\Sigma$. By the above discussion, there exists $c'' > 0$ such that $\sigma(g'), \sigma(\bar{u}'), \sigma(\bar{u}'g') < c''\log(N)$ for g and $\bar{u}$ as in the lemma.

Let c' be the c' associated to $c = c''$ for $\gamma_{X''}$ and X''_Σ. This c' is what we need for $\underline{\gamma_{X''}}$ and $\underline{X_\Sigma}''$. The reverse direction is similar. This proves (4).

We go back to the proof of the lemma. We only deal with the case when x is in the center, the other cases follow from the same method and the calculation is much easier. In this case, $X = X''$. We replace X'' by X for the rest of the proof. Since T is split, $M_\sharp = T$. We may choose $P_\sharp = M_\sharp N_\sharp \in \mathcal{P}(M_\sharp)$ and only consider those $a \in A_T(F)$ with $H_{M_\sharp}(a) \in CL(\mathfrak{a}^+_{P_\sharp})$. Then we must have $P_\sharp \subset Q$. By conjugating by a Weyl element w, we may assume that $P_\sharp \subset \bar{P}$ is the lower Borel subgroup. Note that when we conjugate by w, we just need to make the following transfers: $X \to wXw^{-1}$, $\gamma_X \to w\gamma_X$, $a \to waw^{-1}$, $\bar{u} \to w\bar{u}w^{-1}$ and $g \to wg$. This is allowable by (4). We note that although in (3) we reduce to the case where T split, it still matters whether we are starting from the split case or the nonsplit case since the definition of κ_N really depends on it. If we are in the nonsplit case, we can make $\bar{P} \subset Q$ since $\bar{P}$ is the minimal parabolic subgroup in this case; but this is not possible in the split case since $\bar{P}$ will no longer be the minimal parabolic subgroup.

For $X = \mathrm{diag}(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathfrak{t}_{0,reg}(F)$, if $\mid x_2 - x_1 \mid_F \geq \max\{\mid x_3 - x_4 \mid_F, \mid x_5 - x_6 \mid_F\}$, define

$$X'_\Sigma = \begin{pmatrix} X_1 & 0 & 0 \\ aI_2 & X_2 & 0 \\ 0 & bI_2 & X_3 \end{pmatrix}$$

where we define $X_1 = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$, $X_2 = \begin{pmatrix} x_3 + m & 1 \\ -m^2 + Bm & x_4 - m \end{pmatrix}$, and $X_3 = \begin{pmatrix} x_5 + n & -n^2 + Cn \\ 1 & x_6 - m \end{pmatrix}$ with

$$m = \frac{A+B+C}{2} \cdot \frac{A+B-C}{2A}, \quad n = \frac{A+B+C}{2} \cdot \frac{A+C-B}{2A},$$

where $A = x_2 - x_1$, $B = x_4 - x_3$, and $C = x_6 - x_5$. Then the map $X \to X'_\Sigma$ satisfies condition (1). (Note that we assume $\mid A \mid \geq \max\{\mid B \mid, \mid C \mid\}$.) We can find an element $p_X \in \bar{P}$ of the form $p_X = \bar{u}_X m_X$ such that $p_X X'_\Sigma p_X^{-1} = X$ where

$$m_X = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \in M, \bar{u}_X \in \bar{U}.$$

It follows that $m_X \mathrm{diag}(X_1, X_2, X_3) m_X^{-1} = X$. So we can choose

$$m_1 = I_2, m_2^{-1} = \begin{pmatrix} 1 & 1 \\ -m & B-m \end{pmatrix}, m_3^{-1} = \begin{pmatrix} 1 & 1 \\ -n & C-n \end{pmatrix}.$$

Similarly, we can define m_X and X'_Σ for the case when $\mid x_3 - x_4 \mid_F \geq \max\{\mid x_1 - x_2 \mid_F, \mid x_5 - x_6 \mid_F\}$ or $\mid x_5 - x_6 \mid_F \geq \max\{\mid x_1 - x_2 \mid_F, \mid x_3 - x_4 \mid_F\}$.

Now by adding polynomials $x_1 - x_2$, $x_3 - x_4$ and $x_5 - x_6$ into the set $\mathcal{Q}_T$, for any $X \in \omega_T \cap \mathfrak{t}^0(F)[> N^{-b}]$, we have $\sigma(m) \ll \log(N)$. Applying Proposition 2.4.2 again, we know that p_X, X'_Σ satisfy Conditions (1) and (2). In fact, here we know that $\sigma_T(p_X) \ll \log(N)$ and $\sigma(m_X) \ll \log(N)$ for $X \in \omega_T \cap \mathfrak{t}^0(F)[> N^{-b}]$, these force $\sigma(\bar{u}_X) \ll \log(N)$. Now by (4), it is enough to prove this Lemma for p_X, X'_Σ.

We will only deal with the case when $\mid x_2 - x_1 \mid_F \geq \max\{\mid x_3 - x_4 \mid_F, \mid x_5 - x_6 \mid_F\}$, the rest cases follow from a similar calculation. Applying the Bruhat decomposition, we have

$$m_2^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & B-m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{m}{m-B} & 1 \end{pmatrix} = b_{X,2} w_{X,2}.$$

Similarly we can decompose m_3 and m_1 in this way. Let

$$b_X = \mathrm{diag}(b_{X,1}, b_{X,2}, b_{X,3}), w_X = \mathrm{diag}(w_{X,1}, w_{X,2}, w_{X,3}).$$

By adding some more polynomials on $\mathcal{Q}_T$, we may still assume that $\sigma(w_X) \ll \log(N)$. (Note that $\frac{m}{m-B}$ and $\frac{n}{n-C}$ are rational functions of the x_i's.) It follows that $\sigma(b_X) \ll \log(N)$. Now we can write

$$p_X^{-1} = b_X w_X (\bar{u}_X)^{-1} = b_X v_X$$

for some $v_X = w_X(\bar{u}_X)^{-1} \in U_\sharp(F)$, and we still have $\sigma(b_X), \sigma(v_X) \ll \log(N)$. Since $P_\sharp \subset Q$, we can write $v_X = n_X u_X$ where $n_X \in U_\sharp(F) \cap L(F)$ and $u_X \in U_Q(F)$. Then we have

$$\begin{aligned} v_X a\bar{u}g &= n_X u_X a\bar{u}g = n_X a\bar{u}g \cdot (g^{-1}\bar{u}^{-1}a^{-1}u_X a\bar{u}g) \\ &= a((a^{-1}n_X a)^{-1}\bar{u}(a^{-1}n_X a)) \cdot (a^{-1}n_X^{-1}ag) \cdot (g^{-1}\bar{u}^{-1}a^{-1}u_X a\bar{u}g) \\ &= a\bar{u}'g'k. \end{aligned}$$

For all $a \in A_T(F)$ with $\inf\{\alpha(H_{M_\sharp}(a)) \mid \alpha \in \Sigma_Q^+\} > c_4 \log(N)$ for some $c_4 > 0$ large, $a^{-1}u_X a - 1$ is very close to zero. Hence we can make

$$k = g^{-1}\bar{u}^{-1}a^{-1}u_X a\bar{u}g \in K$$

for all $\sigma(g), \sigma(\bar{u}) < c\log(N)$. Since κ_N is right K-invariant, we have

$$\begin{aligned}\kappa_N(p_X^{-1}a\bar{u}g) = \kappa_N(b_X v_X a\bar{u}g) = \kappa_N(b_X a\bar{u}'g'),\\ \kappa_N(p_X^{-1}ag) = \kappa_N(b_X v_X ag) = \kappa_N(b_X ag').\end{aligned} \tag{11.32}$$

Also since $H_{M_\sharp}(a) \in CL(\mathfrak{a}_{P_\sharp}^+)$, $a^{-1}n_X a$ is a contraction of n_X, and hence we still have $\sigma(\bar{u}'), \sigma(g') \ll \log(N)$.

If we are in the non-split case, then we have already make $\bar{P} \subset Q$, and hence $U_{\bar{Q}} \subset U$. So the $\bar{u}'$ of the first equation in (11.32) can be moved to the very left via the a-conjugation and the b_X-conjugation. Then we can eliminate it by using left U-invariance property of κ_N. This proves the Lemma.

If we are in the split case, we may assume that $\bar{u}' \in U_{\bar{Q}}(F) \cap M(F)$ since the rest part can be switched to the front via the a-conjugation and the b_X-conjugation, and then be eliminated by the left U-invariance property of κ_N. Let $g' = u'm'k'$ be the Iwasawa decomposition with $u' \in U(F), m' \in M(F)$ and $k' \in K$. Then $\sigma(m') \leq c_0 \log(N)$ for $c_0 = lc$ where l is a fixed constant only depends on G. (Here we use the fact that the Iwasawa decomposition preserves the norm up to a bounded constant which only depends on the group and the parabolic subgroup.) We can eliminate u' and k' by the left U-invariance and right K-invariance properties of κ_N. Now applying the Iwasawa decomposition again, we can write $m' = b'k'$ with b' upper triangle. By the same reason, we have $\sigma(b') \leq c_1 \log(N)$ for some $c_1 = l'c_0 = ll'c$. Again by the right K-invariance property of κ_N, we can eliminate k'. b' can be absorbed by a and $\bar{u}'$. After this process, we will still have the majorization for $\bar{u}'$ (i.e. $\sigma(\bar{u}') \ll \log(N)$), and we will still have $\alpha(H_{M_\sharp}(a)) > c'' \log(N)$ for all $\alpha \in \Sigma_Q^+$, here $c'' = c' - c_1$. So we may assume that $m' = 1$. In this case, we have

$$\kappa_N(b_X ag') = \kappa_N(b_X a), \kappa_N(a\bar{u}'g') = \kappa_N(b_X a\bar{u}').$$

Now let $b_X a = \mathrm{diag}(l_1, l_2, l_3)$ and $b_X a\bar{u}' = \mathrm{diag}(l_1', l_2', l_3')$ where l_i and l_i' are all upper triangle 2-by-2 matrices. Since $\bar{u}'$ is an unipotent element and $\sigma(\bar{u}') \ll \log(N)$, $l_i' = l_i n_i$

for some unipotent element n_i with $\sigma(n_i) \ll \log(N)$. Then we know for any $1 \le i, j \le 3$, $l_i^{-1} l_j = \begin{pmatrix} a & x \\ 0 & c \end{pmatrix}$, and $(l_i')^{-1} l_j' = n_i^{-1} \begin{pmatrix} a & x \\ 0 & c \end{pmatrix} n_j$. Since in the definition of κ_N for the split case (as in (5.4)), we do allow the unipotent part to be bounded by $(1+\epsilon)N$ while the diagonal part is bounded by N. Now those n_i's will only add something majorized by $N + C\log(N)$ on the unipotent part and not change the semisimple part. So if we take N large so that $\epsilon N > C\log(N)$, we have

$$\kappa_N(b_X a \bar{u}' g') = \kappa_N(b_X a g').$$

This finishes the proof of the split case, and finishes the proof of the Lemma. $\square$

We prove 11.3(6).

For $c_4 > 0$, by 11.3(1), we impose the mirror condition

$$c_4 \log(N) < c_1 \inf\{\alpha(Z_{P_{min}}) \mid \alpha \in \Delta_{min}\}$$

to $Z_{P_{min}}$ to make sure all terms are well defined.

By the same argument as in Section 11.4, we know that the function $\zeta \to \sigma_{M_\sharp}^Q(\zeta, \mathcal{Z}(g))\tau_Q(\zeta - \mathcal{Z}(g)_Q)$ is invariant under $g \to \bar{u}g$. Therefore

$$\kappa_{N,X''}(Q, \bar{u}g) - \kappa_{N,X''}(Q, g) \tag{11.33}$$

$$= \int_{Z_G \backslash A_T(F)} \sigma_{M_\sharp}^Q(H_{M_\sharp}(a), \mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q)(\kappa_N(\gamma_{X''}^{-1} a\bar{u}g) - \kappa_N(\gamma_{X''}^{-1} ag))du.$$

Let $c = c_4$ be as in Lemma 11.5.1. Then we get some $c' > 0$. For $a \in A_T(F)$ with $\sigma_{M_\sharp}^Q(H_{M_\sharp}(a), \mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q) \neq 0$, by the definition of $\sigma_{M_\sharp}^Q$, τ_Q, and the majorization of g, we have

$$\inf\{\alpha(H_{M_\sharp}(a)) \mid \alpha \in \Sigma_Q^+\} - \inf\{\alpha(Z_{P_{min}}) \mid \alpha \in \Delta_{min}\} \gg -\log(N). \tag{11.34}$$

Now choose $c_5 > 0$ such that $c_5 > \frac{c_4}{c_1}$. We also require that

$$\inf\{\alpha(Z_{P_{min}}) \mid \alpha \in \Delta_{min}\} > c_5 \log(N).$$

Combining with (11.34), we have

$$\inf\{\alpha(H_{M_\sharp}(a)) \mid \alpha \in \Sigma_Q^+\} > c' \log(N).$$

We claim that this is the c_5 we need for 11.3(6). In fact, by the discussion above together with Lemma 11.5.1, we know that for g and $\bar{u}$ as in 11.3(6), $\kappa_N(\gamma_{X''}^{-1} a\bar{u}g) = \kappa_N(\gamma_{X''}^{-1} ag)$ whenever $\sigma_{M_\sharp}^Q(H_{M_\sharp}(a), \mathcal{Z}(g))\tau_Q(H_{M_\sharp}(a) - \mathcal{Z}(g)_Q) \neq 0$. This means that the right hand side of (11.33) equals zero. Hence $\kappa_{N,X''}(Q, \bar{u}g) - \kappa_{N,X''}(Q, g) = 0$. This finishes the proof of 11.3(6).

11.6 Principal Proposition

Proposition 11.6.1. *There exists $N_1 > 0$ such that for $N > N_1$, $X \in \mathfrak{t}_0(F)[> N^{-b}]$, and $x \in H_{ss}(F)$ elliptic, we have*

$$\int_{A_T(F)Z_{G_x}(F)\backslash G(F)} ({}^g f_{x,\omega})\hat{}(X)\kappa_{N,X''}(g)dg = \nu(A_T)\nu(Z_{G_x})\theta_{f,x,\omega}^\sharp(X).$$

Proof. By Proposition 11.3.1, we can replace the function $\kappa_{N,X''}$ by the function $\tilde{v}(g, Y_{P_{min}})$ in the integral above. Then by the computation of $\tilde{v}(g, Y_{P_{min}})$ in [Ar91], together with the same argument as in Proposition 10.9 of [W10], as $Y_{P_{min}}$ goes to infinity, the integral equals

$$(-1)^{a_{M_\sharp} - a_G} \Sigma_{Q \in \mathcal{F}(M_\sharp)} c'_Q I(Q) \tag{11.35}$$

where c'_Q are some constant numbers with $c'_G = 1$, and

$$I(Q) = \int_{Z_{G_x}(F)A_T(F)\backslash G(F)} {}^g f_{x,\omega}^\sharp(X) v_{M_\sharp}^Q(g)dg. \tag{11.36}$$

If $Q = LU_Q \neq G$, we can decompose the integral in (11.36) as $\int_{Z_{G_x}(F)A_T(F)\backslash L(F)}$, $\int_{K_{min}}$ and $\int_{U_Q(F)}$. Since $v_{M_\sharp}^Q(g)$ is $U_Q(F)$-invariant, the inner integral becomes

$$\int_{U_Q(F)} {}^{ulk} f_{x,\omega}^\sharp(X)du.$$

By Lemma 3.7.2, this is zero because f is strongly cuspidal. Therefore

$$I(Q) = 0. \tag{11.37}$$

For $Q = G$, we can replace the integral on $Z_{G_x}(F)A_T(F)\backslash G(F)$ by $T(F)\backslash G(F)$ and multiply it by $meas(T(F)/Z_{G_x}A_T(F))$. Then we get

$$I(G) = meas(T(F)/Z_{G_x}A_T(F))D^{G_x}(X)^{1/2} J_{M_\sharp,x,\omega}^\sharp(X, f) \tag{11.38}$$

where $J^{\sharp}_{M_\sharp,x,\omega}(X,f)$ is defined in (3.17).

Now combining (11.35), (11.37) and (11.38), together with the definition of $\theta^{\sharp}_{f,x,\omega}$ (as in (3.19)) and the fact that

$$\nu(T)meas(T(F)/Z_{G_x}A_T(F)) = \nu(A_T)\nu(Z_{G_x}),$$

we have

$$\int_{A_T(F)Z_{G_x}\backslash G(F)} ({}^g f_{x,\omega})\hat{}(X)\kappa_{N,X''}(g)dg = \nu(A_T)\nu(Z_{G_x})\theta^{\sharp}_{f,x,\omega}(X).$$

This finishes the proof of the Proposition. □

Finally, for $x \in H_{ss}(F)$ elliptic, let

$$\begin{aligned} J_{x,\omega}(f) =& \Sigma_{T\in T(G_x)} \mid W(G_x,T) \mid^{-1} \nu(Z_{G_x}) \\ &\times \int_{\mathfrak{t}'(F)\times(\mathfrak{t}'')^0(F)} D^{G_x}(X'')^{1/2}\theta^{\sharp}_{f,x,\omega}(X)dX. \end{aligned} \tag{11.39}$$

Proposition 11.6.2. *The integral in* (11.39) *is absolutely convergent, and we have*

$$\lim_{N\to\infty} I_{x,\omega,N}(f) = J_{x,\omega}(\theta,f).$$

Proof. The proof for the first part is the same as Lemma 10.10(1) in [W10]. For the second part, by Lemma 11.1.2, it is enough to consider $\lim_{N\to\infty} I^*_{x,\omega,N}(\theta,f)$. Then the proposition just follows from Proposition 11.6.1 together with (11.1). □

Chapter 12

The Proof of the Trace Formula

In this chapter, we are going to prove the geometric side of the trace formula. In Section 12.1, by applying the computation in the previous chapter, we are going to compute the limit $lim_{N\to\infty}I_N(f)$ in terms of the distribution $\hat{\theta}_f$ for the Lie algebra case. Then in Section 12.2, we are going to prove the Lie algebra version of the trace formula based on a hypothesis. In Section 12.3, we will finish the proof of the trace formula based on the trace formula of the reduced model (i.e. the Whittaker model). Finally, in Section 12.4, we prove the trace formula for the reduced model.

12.1 Calculation of $lim_{N\to\infty}I_N(f)$: the Lie Algebra Case

If $f \in C_c^\infty(\mathfrak{g}_0(F))$ is a strongly cuspidal function, we define

$$J(f) = \Sigma_{T\in T(G)} \mid W(G,T) \mid^{-1} \int_{\mathfrak{t}^0(F)} D^G(X)^{1/2}\hat{\theta}_f(X)dX. \tag{12.1}$$

Lemma 12.1.1. *The integral in* (12.1) *is absolutely convergent and*

$$\lim_{N\to\infty} I_N(f) = J(f).$$

Proof. The first part is similar to the first part of Proposition 11.6.2. For the second part, let $\omega \subset \mathfrak{g}_0(F)$ be a good neighborhood of 0. Suppose that $Supp(f) \subset \omega$. Then we can relate f to a function Φ on $Z_G(F)\backslash G(F)$ which is supported on $Z_G(F)\exp(\omega)$. By Proposition 9.3.4, we know $I_N(f) = I_N(\Phi)$. Then by Proposition 11.6.2, applying to the function Φ and $x = 1$, we have $\lim_{N\to\infty} I_N(f) = J_{1,\omega}(\Phi)$. By Proposition 3.7.4,

$\theta^{\sharp}_{\Phi,1,\omega}$ is the partial Fourier transform of $\theta_{\Phi,1,\omega} = \theta_f$. But for $x = 1$, partial Fourier transform is just the full Fourier transform. Thus $\theta^{\sharp}_{\Phi,1,\omega} = \hat{\theta}_f$. Also we know that $\nu(Z_{G_x}) = \nu(Z_G) = 1$. Therefore

$$\lim_{N\to\infty} I_N(f) = J_{1,\omega}(\Phi) = J(f).$$

This proves the Lemma for those f whose support is contained in ω.

In general, replacing (a,b) in the definition of ξ (as in (5.1)) by $(\lambda a, \lambda b)$ for some $\lambda \in F^{\times}$, we get a new character ξ', and let $f' = f^{\lambda}$. Then for $Y \in \mathfrak{h}(F)$, we have

$$(f')^{\xi'}(Y) = \mid \lambda \mid_F^{-\dim(U)} f^{\xi}(\lambda Y).$$

This implies

$$I_{\xi',N}(f') = \mid \lambda \mid_F^{-\dim(U)-\dim(H/Z_H)} I_{\xi,N}(f). \tag{12.2}$$

On the other hand, we know

$$\begin{aligned} \hat{\theta}_{f'}(X) &= \mid \lambda \mid_F^{-\dim(G/Z_G)} \hat{\theta}_f(\lambda^{-1}X), \\ D^G(\lambda X)^{1/2} &= \mid \lambda \mid_F^{\delta(G)/2} D^G(X)^{1/2}, \end{aligned}$$

and $\mathfrak{t}^0(F)$ will not change under this transform. By changing of variable in (12.1), (Note that this is allowable since $\mathfrak{t}^0(F)$ is invariant under scalar in the sense that for $t \in \mathfrak{t}^0(F), \lambda \in F^{\times}$, we have $\lambda t \in \mathfrak{t}^0(F)$, see Remark 10.4.3) we have

$$J_{\xi'}(f') = \mid \lambda \mid_F^{-\dim(G/Z_G)+\dim(T/Z_G)+\delta(G)/2} J_{\xi}(f). \tag{12.3}$$

Because

$$-\dim(G/Z_G) + \dim(T/Z_G) + \delta(G)/2 = -\dim(U) - \dim(H/Z_H) = -15,$$

together with (12.2) and (12.3), we know that $\lim_{N\to\infty} I_{\xi,N}(f) = J_{\xi}(f)$ if and only if $\lim_{N\to\infty} I_{\xi',N}(f') = J_{\xi'}(f')$. Then for any f, we can choose λ such that $Supp(f') \subset \omega$. Applying the first part of the proof to f', we get $\lim_{N\to\infty} I_{\xi',N}(f') = J_{\xi'}(f')$, which implies $\lim_{N\to\infty} I_{\xi,N}(f) = J_{\xi}(f)$. This finishes the proof of the Lemma. $\square$

12.2 A Premier Result

During this section, consider the following hypothesis.

Hypothesis: For every strongly cuspidal $f \in C_c^\infty(\mathfrak{g}_0(F))$ whose support dose not contain any nilpotent element, we have

$$\lim_{N\to\infty} I_N(f) = I_{geom}(f).$$

In this section, we will prove the following proposition.

Proposition 12.2.1. *If the above hypothesis holds, we have*

$$\lim_{N\to\infty} I_N(f) = I_{geom}(f)$$

for every strongly cuspidal $f \in C_c^\infty(\mathfrak{g}_0(F))$.

In order to prove the above proposition, consider the following morphism:

$$f \to E(f) = \lim_{N\to\infty} I_N(f) - I_{geom}(f) = J(f) - I_{geom}(f) \tag{12.4}$$

defined on the space of strongly cuspidal functions $f \in C_c^\infty(\mathfrak{g}_0(F))$. This is obviously a linear map.

Lemma 12.2.2. *The map E is a scalar multiple of the morphism $f \to c_{\theta_f,\mathcal{O}}$ where $\mathcal{O}$ is the regular nilpotent orbit of $\mathfrak{g}(F)$. In particular, $E = 0$ if $G = GL_3(D)$.*

Proof. We first prove:

(1) $E(f) = 0$ if $c_{\theta_f,\mathcal{O}} = 0$ for every $\mathcal{O} \in Nil(\mathfrak{g}(F))$.

Suppose that $c_{\theta_f,\mathcal{O}} = 0$ for every $\mathcal{O} \in Nil(\mathfrak{g}(F))$. We can find a G-domain ω in $\mathfrak{g}_0(F)$, which has compact support modulo conjugation and contains 0, such that $\theta_f(X) = 0$ for every $X \in \omega$. Let $f' = f1_\omega$ and $f'' = f - f'$. Then these two functions are also strongly cuspidal. The support of f'' does not contain nilpotent elements. By the hypothesis, we know that $E(f'') = 0$.

On the other hand, since $\theta_f(X) = 0$ for every $X \in \omega$, we have $\theta_{f'} = 0$ and $\hat{\theta}_{f'} = 0$. By the definition of $I_{geom}(f)$ and $J(f)$, we know that $J(f') = 0 = I_{geom}(f')$. Hence

$E(f) = E(f') + E(f'') = 0$. This proves (1).

Now for $\lambda \in (F^\times)^2$, let $f' = f^\lambda$. We have $\theta_{f'} = (\theta_f)^\lambda$. For $\mathcal{O} \in Nil(\mathfrak{g}(F))$, by (3.4), we have

$$c_{\theta_{f'},\mathcal{O}} =\mid \lambda \mid_F^{-\dim(\mathcal{O})/2} c_{\theta_f,\mathcal{O}}. \tag{12.5}$$

We then show:

(2) $E(f') =\mid \lambda \mid_F^{-\delta(G)/2} E(f) =\mid \lambda \mid_F^{-15} E(f)$

By (12.3), we have

$$J(f') =\mid \lambda \mid_F^{-15} J(f). \tag{12.6}$$

Now for $I_{geom}(f)$, let $T \in \mathcal{T}$ as in Section 5.2. The expression for $I_{geom}(f)$ related to T is

$$\int_{t_0(F)} c_f(Y) D^H(Y)\Delta(Y)dY. \tag{12.7}$$

If $T = \{1\}$, (12.7) $= c_f(0)$ is the germ associated to the unique regular nilpotent orbit of $\mathfrak{g}(F)$. By (3.4), we have

$$c_{f'}(0) =\mid \lambda \mid_F^{-\delta(G)/2} c_f(0) =\mid \lambda \mid_F^{-15} c_f(0).$$

If $T = T_v$ for some $v \in F^\times/(F^\times)^2, v \neq 1$ as in Section 5.2, the nilpotent orbit associated to c_f is the unique regular nilpotent orbit $\mathcal{O}_v$ of $GL_3(F_v)$, which is of dimension 12. By (3.4) again, we have

$$c_{f'}(X) =\mid \lambda \mid_F^{-6} c_f(\lambda X).$$

Moreover, $D^H(\lambda^{-1}X) =\mid \lambda \mid_F^{-2}$ since $\dim(\mathfrak{h}) - \dim(\mathfrak{h}_x) = 2$, and $\Delta(\lambda^{-1}X) =\mid \lambda \mid_F^{-6} \Delta(X)$ since $\dim(\mathfrak{u}) - \dim(\mathfrak{u}_x) = 6$. Therefore by changing variable $X \to \lambda^{-1}X$, we have

$$\int_{t_0(F)} c_{f'}(Y) D^H(Y)\Delta(Y)dY =\mid \lambda \mid_F^{b} \int_{t_0(F)} c_f(Y) D^H(Y)\Delta(Y)dY \tag{12.8}$$

where $b = -6 - 2 - 6 - \dim(\mathfrak{t}_0) = -15$. Combining (12.7) and (12.8), we have

$$I_{geom}(f') =\mid \lambda \mid_F^{-15} I_{geom}(f). \tag{12.9}$$

Then (2) just follows from (12.6) and (12.9).

Now (1) tells us that E is a linear combination of $c_{\theta_f,\mathcal{O}}$ for $\mathcal{O} \in Nil(\mathfrak{g}(F))$. We know that $\dim(\mathcal{O}) \leq 30$ and the equality holds if and only if $G = GL_6(F)$ and $\mathcal{O}$ is regular. Hence the Lemma follows from (2) and (12.5). $\square$

In particular, by the lemma above, we have proved Proposition 12.2.1 for $G = GL_3(D)$. Now we are going to prove the case when $G = GL_6(F)$.

By the discussion above, in this case, $E(f) = c_{reg}c_{\theta_f,\mathcal{O}_{reg}}$ for some complex number c_{reg}. It is enough to show that $c_{reg} = 0$. Our method is to find some special f such that $E(f) = 0$ and $c_{\theta_f,\mathcal{O}_{reg}} = 1$. This will implies that $c_{reg} = 0$. The way to find this f is due to Waldspurger, see [W10].

By 6.3(3) and 11.5 of [W10], for $T \in T(G)$ (here $T(G)$ is the set of equivalent classes of maximal subtorus of $G(F)$) and $X \in \mathfrak{t}_0(F) \cap \mathfrak{g}_{reg}(F)$, we can construct a neighborhood ω_X of X in $\mathfrak{t}_0(F)$ and a strongly cuspidal function $f[X] \in C_c^\infty(\mathfrak{g}_0(F))$ satisfy the following conditions:

1. For $T' \in T(G)$ with $T' \neq T$, the restriction of $\hat{\theta}_{f[X]}$ to $\mathfrak{t}'_0(F)$ is zero.
2. For every locally integrable function φ on $\mathfrak{t}_0(F)$ which is invariant under the conjugation of Weyl group, we have
$$\int_{\mathfrak{t}_0(F)} \varphi(X') D^G(X')^{1/2} \hat{\theta}_{f[X]}(X') dX' = \mid W(G,T) \mid meas(\omega_X)^{-1} \int_{\omega_X} \varphi(X') dX'$$
3. For every $\mathcal{O} \in Nil(\mathfrak{g})$, we have
$$c_{\theta_{f[X]},\mathcal{O}} = \Gamma_{\mathcal{O}}(X)$$
where $\Gamma_{\mathcal{O}}(X)$ is the Shalika germ defined in Section 2.5.

Now let T_d be the unique split torus of $T(G)$. This is possible since we are in the split case now. Fix $X_d \in \mathfrak{t}_{d,0}(F) \cap \mathfrak{g}_{reg}(F)$. Then we can find ω_{X_d} and $f[X_d]$ as above. Let $f = f[X_d]$. By condition (3) above and Lemma 11.4(i) of [W10], we know that $c_{\theta_f,\mathcal{O}_{reg}} = 1$. This implies
$$E(f) = c_{reg}. \tag{12.10}$$

Now by condition (1) above, we know that each components of the summation in $I_{geom}(f)$ is 0 for $T \in \mathcal{T}$ with $T \neq \{1\}$. Then by applying condition (3) above and Lemma 11.4(i) of [W10] again, we have

$$I_{geom}(f) = c_{\theta_f,\mathcal{O}_{reg}} = 1. \tag{12.11}$$

On the other hand, by condition (1) and (2),

$$\begin{aligned} J(f) &= \Sigma_{T\in T(G)} \mid W(G,T) \mid^{-1} \int_{\mathfrak{t}^0(F)} D^G(X)^{1/2}\hat{\theta}_f(X)dX \\ &=\mid W(G,T_d) \mid^{-1} \int_{\mathfrak{t}_{d,0}(F)} D^G(X)^{1/2}\hat{\theta}_f(X)dX \\ &= meas(\omega_{X_d})^{-1} meas(\omega_{X_d}) = 1. \end{aligned} \tag{12.12}$$

Here we use the fact that $(\mathfrak{t}_d)^0(F) = \mathfrak{t}_{d,0,reg}(F)$, which has been proved in the proof of Lemma 11.5.1.

Now combining (12.10), (12.11) and (12.12), we have

$$c_{reg} = E(f) = I_{geom}(f) - J(f) = 1 - 1 = 0.$$

This finishes the proof of Proposition 12.2.1.

12.3 Proof of the Trace Formula

Consider the following four assertions:

$(th)_G$: For every strongly cuspidal function $f \in C_c^\infty(Z_G(F)\backslash G(F))$, we have $\lim_{N\to\infty} I_N(f) = I_{geom}(f)$.

$(th')_G$: For every strongly cuspidal function $f \in C_c^\infty(Z_G(F)\backslash G(F))$ whose support does not contain any unipotent element, we have $\lim_{N\to\infty} I_N(f) = I_{geom}(f)$.

$(th)_{\mathfrak{g}}$: For every strongly cuspidal function $f \in C_c^\infty(\mathfrak{g}_0(F))$, we have $\lim_{N\to\infty} I_N(f) = I_{geom}(f)$.

$(th')_{\mathfrak{g}}$: For every strongly cuspidal function $f \in C_c^\infty(\mathfrak{g}_0(F))$ whose support does not contain any nilpotent element, we have $\lim_{N\to\infty} I_N(f) = I_{geom}(f)$.

Lemma 12.3.1. *The assertion* $(th)_G$ *implies* $(th)_{\mathfrak{g}}$. *The assertion* $(th')_G$ *implies* $(th')_{\mathfrak{g}}$.

Proof. Suppose that $(th)_G$ holds. For any strongly cuspidal function $f \in C_c^\infty(\mathfrak{g}_0(F))$, we need to show $E(f) = 0$. In the proof of Lemma 12.2.2, we have proved that $E(f) = \mid \lambda \mid_F^{15} E(f^\lambda)$. So by changing f to f^λ, we may assume that the support of f is contained in a good neighborhood ω of 0 in $\mathfrak{g}_0(F)$. Same as in Lemma 12.1.1, we can construct a strongly cuspidal function $F \in C_c^\infty(Z_G(F)\backslash G(F))$ such that $J(f) = J_{1,\omega}(F)$ and $I_{geom}(f) = I_{1,\omega}(F)$. By Propositions 9.3.4, 9.4.1, and 11.6.2, we have $J_{1,\omega}(F) = \lim_{N\to\infty} I_N(F)$ and $I_{1,\omega}(F) = I_{geom}(F)$. By $(th)_G$, we have $I_{geom}(F) = J_{1,\omega}(F)$, which implies $E(f) = 0$.

The proof of the second part is similar to the proof of the first part: we only need to add the fact that if the support of f does not contain any nilpotent element, then the support of F does not contain any unipotent element. □

We first prove $(th')_G$.

Proof. Let $f \in C_c^\infty(Z_G(F)\backslash G(F))$ be a strongly cuspidal function whose support does not contain any unipotent element. For $x \in G_{ss}(F)$, let ω_x be a good neighborhood of 0 in $\mathfrak{g}_x(F)$, and let $\Omega_x = (x\exp(\omega_x))^G \cdot Z_G$. We require that ω_x satisfies the following conditions:

1. If x belongs to the center, since f is $Z_G(F)$-invariant, we may assume that $x = 1$. We require that $\Omega_x \cap Supp(f) = \Omega_1 \cap Supp(f) = \emptyset$. This is possible since the support of f does not contain any unipotent element.

2. If x is not conjugated to any element in $H(F)$, choose ω_x satisfying the condition in Section 9.1.

3. If x is conjugated to a non-elliptic element $x' \in H_{ss}(F)$, choose ω_x satisfying the condition in Section 9.2.

4. If x is conjugated to an elliptic element $x' \in H_{ss}(F)$ not in the center, we choose a good neighborhood $\omega_{x'}$ of 0 in $\mathfrak{g}_{x'}(F)$ as in Section 9.3, and let ω_x be the image of $\omega_{x'}$ by conjugation. Moreover, we choose $\omega_{x'}$ small enough such that $\Omega_{x'}$ does not contain split element.

Then we can choose a finite set $\mathcal{X} \subset G_{ss}(F)$ such that $f = \Sigma_{x\in\mathcal{X}} f_x$ where f_x is the product of f and the characteristic function on Ω_x. Since $\lim_{N\to\infty} I_N(f)$ and $I_{geom}(f)$ are linear functions on f, we may just assume that $f = f_x$.

If $x = 1$, by the choice of Ω_1 we know that $f = 0$, and the assertion is trivial.

If x is not conjugated to an element of $H(F)$, then the assertion follows from the choice of Ω_x and the same argument as in Section 9.1.

If x is conjugated to a non-elliptic element of $H(F)$, then the assertion follows from the choice of Ω_x and the same argument as in Section 9.2.

If x is conjugated to an elliptic element of H. By Propositions 9.3.4 and 9.4.1, it is enough to prove

$$\lim_{N\to\infty} I_{x,\omega,N}(f) = I_{x,\omega}(f). \tag{12.13}$$

Now we can decompose $\theta_{f,x,\omega}$ as

$$\theta_{f,x,\omega}(X) = \Sigma_{b\in B}\theta'_{f,b}(X')\theta''_{f,b}(X'') \tag{12.14}$$

where B is a finite index set, and for every $b \in B$, $\theta'_{f,b}(X')$ (resp. $\theta''_{f,b}(X'')$) is a quasi-character on $\mathfrak{g}'_x(F)$ (resp. $\mathfrak{g}''(F)$). By Proposition 6.4 of [W10], for every $b \in B$, we can find $f''_b \in C_c^\infty(\mathfrak{g}''(F))$ strongly cuspidal such that $\theta''_{f,b}(X'') = \theta_{f''_b}$. Then by the definition of $I_{x,\omega}(f)$ (as in (9.24)), we have

$$I_{x,\omega}(f) = \Sigma_{b\in B} I'(b) I_{geom}(f''_b)$$

where

$$I'(b) = \nu(Z_{G_x}) \int_{\mathfrak{g}'_x(F)} \theta'_{f,b}(X')dX',\ I_{geom}(f''_b) = c_{\theta''_{f,b},\mathcal{O}}(1)$$

with $\mathcal{O}$ be the unique regular nilpotent orbit in $\mathfrak{g}''(F)$. Here we use the fact that the only torus in $\mathcal{T}_x$ is Z_{G_x}, which implies that $\nu(T) = \nu(Z_{G_x})$ and $D^{H_x}(X) = \Delta''(X) = 1$ for all $X \in \mathfrak{t}_0(F)$.

On the other hand, by Proposition 11.6.2, we have

$$\lim_{N\to\infty} I_{x,\omega,N}(f) = J_{x,\omega}(f) = \Sigma_{b\in B} I'(b) J(f''_b)$$

where

$$J(f''_b) = \Sigma_{T\in T(G_x)} \mid W(G_x, T) \mid^{-1} \int_{(\mathfrak{t}'')^0(F)} D^{G_x}(X)^{1/2} \hat{\theta}_{f''_b}(X)dX.$$

In order to prove (12.13), we only need to show that $I_{geom}(f_b'') = J(f_b'')$. This is just the Lie algebra version of the trace formula for the model

$$(G_x, U_x),$$

which is just the Whittaker model of $\mathrm{GL}_3(F_v)$. The proof is very similar to the Ginzburg-Rallis model case, we will prove it in the next section. □

Finally we can finish the proof of the trace formula. By Lemma 12.3.1, we only need to prove the group case. We use the same argument as in the proof of $(th')_G$ above, except that in the $x = 1$ case, we don't have $\Omega_1 \cap Supp(f) = \emptyset$. In this case, still by using localization, we can reduce to the Lie algebra case. Now since we have proved $(th')_G$, together with Lemma 12.3.1, we know that $(th')_{\mathfrak{g}}$ holds. Then using Proposition 12.2.1, we get $(th)_{\mathfrak{g}}$, which gives us $(th)_G$. This finishes the proof of the trace formula.

12.4 The proof of $I_{geom}(f_b'') = J(f_b'')$

In this section, we are going to prove

$$I_{geom}(f_b'') = J(f_b''), \tag{12.15}$$

which is the geometric side of the Lie algebra version of the relative trace formula for the Whittaker model of $GL_3(F_v)$. There are two ways to prove it, one is to apply the method we used in previous sections to the Whittaker model case; the other one is to use the spectral side of the trace formula together with the multiplicity formula of the Whittaker model proved by Rodier in [Rod81].

Method I: By the same argument as in Section 12.2, we only need to prove (12.15) for f_b'' whose support does not contain any nilpotent element. Then by changing f_b'' to $(f_b'')^{\lambda}$, we may assume that the function f_b'' is supported on a small neighborhood of 0. Then we can relate f_b'' to a function Φ_x on $G_x(F)/Z_{G_x}(F)$. By the same argument as in the Ginzburg-Rallis model case, we know that in order to prove (12.15), it is enough to prove the geometric side of the local relative trace formula for Φ_x, i.e. $\lim_{N\to\infty} I_N(\Phi_x) = c_{\Phi_x}(1)$. Here $I_N(\Phi_x)$ is defined in the same way as $I_N(f)$ in Section 5.2. In other word, we first integrate over U_x, then integrate on $G_x/U_xZ_{G_x}$. $c_{\Phi_x}(1)$ is the germ of θ_{Φ_x} at 1 associated to the unique regular nilpotent orbit of $\mathfrak{gl}_3(F_v)$.

Since f_b'' does not support on nilpotent element, Φ_x does not support on unipotent element. This implies that $c_{\Phi_x}(1)=0$. On the other hand, since the only semisimple element in U_x is 1, by the same argument as in Section 10.1, the localization of $I_N(\Phi_x)$ at $y\in G_x(F)_{ss}$ is zero if y is not in the center. If we are localizing at 1, since the support of Φ_x does not contain unipotent element, we will still get zero once we choose the neighborhood small enough. Therefore $\lim_{N\to\infty} I_N(\Phi_x)=0=c_{\Phi_x}(1)$, and this proves (12.15).

Method II: Same as in Method I, we reduce to prove the group version of the relative trace formula, i.e. $\lim_{N\to\infty} I_N(\Phi_x)=c_{\Phi_x}(1)$. By applying the same method as in Chapter 4-8, we can prove a spectral expansion of $\lim_{N\to\infty} I_N(\Phi_x)$:

$$\lim_{N\to\infty} I_N(\Phi_x)=\int_{\Pi_{temp}(G_x(F),1)} \theta_\pi(\Phi_x)m'(\bar{\pi})d\pi \tag{12.16}$$

where $\Pi_{temp}(G_x(F),1)$ is the set of all tempered representations of $G_x(F)$ with trivial central character, $d\pi$ is a measure on $\Pi_{temp}(G_x(F),1)$ defined in Section 2.9, $\theta_\pi(\Phi_x)$ is defined in Section 3.5 via the weighted character, and $m'(\bar{\pi})$ is the multiplicity for the Whittaker model (here we are in the GL_n case, all tempered representations are generic, so $m'(\bar{\pi})$ is always 1).

By the work of Rodier, $m'(\bar{\pi})=c_{\bar{\pi}}(1)$ where $c_{\bar{\pi}}(1)$ is the germ of $\theta_{\bar{\pi}}$ at 1 associated to the unique regular nilpotent orbit of $\mathfrak{gl}_3(F_v)$. Therefore (12.16) becomes

$$\lim_{N\to\infty} I_N(\Phi_x)=\int_{\Pi_{temp}(G_x(F),1)} \theta_\pi(\Phi_x)c_{\bar{\pi}}(1)d\pi. \tag{12.17}$$

Finally, as in Proposition 3.5.3, we have

$$\theta_{\Phi_x}=\int_{\Pi_{temp}(G_x(F),1)} \theta_\pi(\Phi_x)\theta_{\bar{\pi}}d\pi.$$

Combining with (12.17), we have $\lim_{N\to\infty} I_N(\Phi_x)=c_{\Phi_x}(1)$ and this proves (12.15).

Chapter 13

The Proof of the Main Theorems

In this chapter, we are going to prove our main theorems (i.e Theorem 1.2.1 and Theorem 1.2.2) for the p-adic case. The key ingredient in the proof is the trace formula we proved in previous chapters. In Section 13.1, by applying the trace formula, we prove a multiplicity formula for the Ginzburg-Rallis model. In Section 13.2, by applying the relations between the distribution characters under the Jacquet-Langlands correspondence in [DKV84], together with the multiplicity formulas, we are able to prove Theorem 1.2.1. In Section 13.3, we are going to prove Theorem 1.2.2.

13.1 The Multiplicity Formulas

Let π be an irreducible tempered representation of $G(F)$ with central character $\eta = \chi^2$. Similar to Section 5.2, we define the geometric multiplicity to be

$$m_{geom}(\pi) = \sum_{T \in \mathcal{T}} |W(H,T)|^{-1} \nu(T) \int_{Z_G(F)\backslash T(F)} c_\pi(t) D^H(t) \Delta(t) \omega(t)^{-1} dt.$$

Here $c_\pi(t) = c_{\theta_\pi}(t)$ is the germ associated to the distribution character θ_π. The multiplicity formula is just

$$m(\pi) = m_{geom}(\pi). \tag{13.1}$$

Let $\pi = I_{\bar{Q}}^G(\tau)$ for some good parabolic subgroup $\bar{Q} = LU_Q$ and some discrete series τ of $L(F)$. In Section 5.4, we have defined the geometric multiplicity $m_{geom}(\tau)$ for the reduced model $(L, R_{\bar{Q}})$. The following lemma tells us the relation between $m_{geom}(\pi)$ and $m_{geom}(\tau)$.

Lemma 13.1.1. *With the notation above, we have*

$$m_{geom}(\pi) = m_{geom}(\tau).$$

Proof. This is a direct consequence of Lemma 3.3.1(2). In fact, if $\bar{Q}$ is of Type I, by applying the lemma, we know that the germs associated to π and τ are the same:

$$D^G(t)^{1/2}c_\pi(t) = D^L(t)^{1/2}c_\tau(t),\ \forall t \in T_{reg}(F), T \in \mathcal{T}.$$

This implies

$$\Delta(t)c_\pi(t) = \Delta_Q(t)c_\tau(t).$$

Hence $m_{geom}(\pi) = m_{geom}(\tau)$. Note that in Section 5.4, we have only defined Δ_Q for the middle model; for the trilinear GL_2 model, Δ_Q is just 1.

If $\bar{Q}$ is of Type II, by applying the lemma, we know that the germ $c_\pi(t)$ is zero for all $t \in T_{reg}(F), T \in \mathcal{T}$ with $t \neq 1$. Therefore we have $m_{geom}(\pi) = c_\pi(1) = 1 = c_\tau(1) = m_{geom}(\tau)$. This proves the lemma. □

The rest of this section is to prove the multiplicity formula (13.1). If π is not a discrete series, with the notation above together with the inductional hypothesis, we have $m(\tau) = m_{geom}(\tau)$. Combining with Corollary 6.6.4 and the lemma above, we have

$$m(\pi) = m(\tau) = m_{geom}(\tau) = m_{geom}(\pi).$$

This proves (13.1).

From now on we assume that π is a discrete series. Combining the trace formula $I_{geom}(f) = I_{spec}(f)$ and Proposition 3.5.3, we have

$$\begin{aligned} &\int_{\Pi'_{temp}(G,\eta^{-1})} \theta_f(\Pi)m(\bar{\Pi})d\Pi + \int_{\Pi^2(G,\eta^{-1})} \theta_f(\Pi)m(\bar{\Pi})d\Pi \\ = &\int_{\Pi'_{temp}(G,\eta^{-1})} \theta_f(\Pi)m_{geom}(\bar{\Pi})d\Pi + \int_{\Pi^2(G,\eta^{-1})} \theta_f(\Pi)m_{geom}(\bar{\Pi})d\Pi. \end{aligned} \tag{13.2}$$

Here as before, $\Pi^2(G,\eta^{-1}) \subset \Pi_{temp}(G,\eta^{-1})$ is the subset consisting of discrete series, and $\Pi'_{temp}(G,\eta^{-1}) = \Pi_{temp}(G,\eta^{-1}) - \Pi^2(G,\eta^{-1})$. By the above discussion, we know the multiplicity formula holds for all $\Pi \in \Pi'_{temp}(G,\eta^{-1})$. Therefore (13.2) becomes

$$\int_{\Pi^2(G,\eta^{-1})} \theta_f(\Pi)m(\bar{\Pi})d\Pi = \int_{\Pi^2(G,\eta^{-1})} \theta_f(\Pi)m_{geom}(\bar{\Pi})d\Pi. \tag{13.3}$$

Now take $f \in C_c^{\infty}(Z_G(F)\backslash G(F), \eta)$ to be the pseudo coefficient of $\bar{\pi}$. This means that $\mathrm{tr}(\bar{\pi}(f)) = 1$ and $\mathrm{tr}(\sigma(f)) = 0$ for all $\sigma \in \Pi_{temp}(G, \eta^{-1})$ with $\sigma \neq \bar{\pi}$. The existence of such an f was proved in Lemma 3.8.1. The lemma also shows that f is strongly cuspidal. For such an f and for any $\Pi \in \Pi^2(G, \eta)$, we have $\theta_f(\Pi) = \mathrm{tr}(\Pi(f))$. Hence it is nonzero if and only if $\Pi = \bar{\pi}$. Therefore (13.3) becomes

$$\theta_f(\bar{\pi})m(\pi) = \theta_f(\bar{\pi})m_{geom}(\pi).$$

Hence $m_{geom}(\pi) = m(\pi)$, and this proves (13.1).

13.2 The Proof of Theorem 1.2.1

In this section, we prove Theorem 1.2.1 by applying the multiplicity formula (13.1) in the previous section. Let $G = GL_6(F)$ and $G_D = GL_3(D)$. Similarly we have $H_0, H_{0,D}, U$ and U_D. Let $\pi, \pi_D, \chi, \omega, \omega_D, \xi$ and ξ_D be the same as in Conjecture 1.1.2. We assume that π is tempered. By (13.1), we have

$$\begin{aligned} m(\pi) &= c_{\theta_\pi, \mathcal{O}_{reg}}(1) + \Sigma_{v \in F^\times/(F^\times)^2, v \neq 1} \mid W(H, T_v) \mid^{-1} \nu(T_v) \\ &\quad \times \int_{Z_H \backslash T_v(F)} \omega^{-1}(t) c_\pi(t) D^H(t) \Delta(t) dt \end{aligned}$$

and

$$\begin{aligned} m(\pi_D) &= \Sigma_{v \in F^\times/(F^\times)^2, v \neq 1} \mid W(H_D, T_v) \mid^{-1} \nu(T_v) \\ &\quad \times \int_{Z_{H_D} \backslash T_v(F)} \omega_D^{-1}(t') c_{\pi_D}(t') D^{H_D}(t') \Delta_D(t') dt. \end{aligned}$$

Here we use t to denote elements in $\mathrm{GL}_6(F)$ and t' to denote elements in $\mathrm{GL}_3(D)$. We can match t and t' via the characteristic polynomial: we write $t \leftrightarrow t'$ if they have the same characteristic polynomial. Since π is tempered, it is generic. So by [Rod81], we have $c_{\theta_\pi, \mathcal{O}_{reg}}(1) = 1$. Also for $v \in F^\times/(F^\times)^2, v \neq 1$, we have

$$\mid W(H_D, T_v) \mid = \mid W(H, T_v) \mid, Z_H = Z_{H_D}.$$

So in order to prove Theorem 1.2.1, we only need to show that for any $v \in F^\times/(F^\times)^2, v \neq 1$, the sum of

$$\int_{Z_H(F) \backslash T_v(F)} \omega^{-1}(t) c_\pi(t) D^H(t) \Delta(t) d_c t$$

and

$$\int_{Z_H(F)\backslash T_v(F)} \omega_D^{-1}(t')c_{\pi_D}(t')D^{H_D}(t')\Delta_D(t')d_ct'$$

equals 0. For $t, t' \in T_v(F)$ regular with $t \leftrightarrow t'$, we have

$$D^H(t) = D^{H_D}(t), \Delta(t) = \Delta_D(t'), \omega(t) = \omega_D(t').$$

Therefore it is enough to show that for any $v \in F^\times/(F^\times)^2, v \neq 1$, and for any $t, t' \in T_v(F)$ regular with $t \leftrightarrow t'$, we have

$$c_\pi(t) + c_{\pi_D}(t') = 0. \tag{13.4}$$

By Section 13.6 of [W10] or Proposition 4.5.1 of [B15], we have

$$c_\pi(t) = D^G(t)^{-1/2}|W(G_t, T_{qs,t}|^{-1} \lim_{x\in T_{qs,t}(F)\to t} D^G(x)^{1/2}\theta_\pi(x)$$

and

$$c_{\pi_D}(t') = D^{G_D}(t)^{-1/2}|W((G_D)_{t'}, T_{qs,t'}|^{-1} \lim_{x'\in T_{qs,t'}(F)\to t} D^{G_D}(x')^{1/2}\theta_{\pi_D}(x')$$

where $T_{qs,t}$ (resp. $T_{qs,t'}$) is a maximal torus contained in the Borel subgroup B_t (resp. $B_{t'}$) of G_t (resp. $(G_D)_{t'}$). Note that if $t, t' \in T_v$ is regular, both G_t and $(G_D)_{t'}$ are isomorphic to $GL_3(F_v)$ which is quasi-split over F. We are able to choose the Borel subgroup B_t (resp. $B_{t'}$). In particular, $|W(G_t, T_{qs,t})|^{-1} = |W((G_D)_t, T_{qs,t})|^{-1}$. Also for those matched $t \leftrightarrow t'$, we have $D^G(t) = D^{G_D}(t)$. And for $x \in T_{qs,t}(F)$ (resp. $x' \in T_{qs,t'}(F)$) sufficiently close to t (resp. t') with $x \leftrightarrow x'$, they are also regular and we have $D^G(x) = D^{G_D}(x')$. Therefore in order to prove (13.4), it is enough to show that for any regular $x \in G(F)$ and $x' \in G_D(F)$ with $x \leftrightarrow x'$, we have

$$\theta_\pi(x) + \theta_{\pi_D}(x') = 0. \tag{13.5}$$

This just follows from the relations of the distribution characters under the Jacquet-Langlands correspondence (see [DKV84]). This proves Theorem 1.2.1

13.3 The Proof of Theorem 1.2.2

Let π be an irreducible tempered representation of $\mathrm{GL}_6(F)$ with trivial central character. Let $\pi = I_{\bar{Q}}^G(\tau)$ for some good parabolic subgroup $\bar{Q} = LU_Q$ and some discrete series τ

of $L(F)$. By our assumptions in Theorem 1.2.2, $\bar{Q}$ can not be of type (6) or type $(4,2)$. Then there are two possibilities: $\bar{Q}$ is of type $(2,2,2)$ or $\bar{Q}$ is of Type II.

If $\bar{Q}$ is of type $(2,2,2)$. By a similar argument as in Section 7.3, we have $\epsilon(1/2,\pi,\wedge^3)=\epsilon(1/2,\tau)$. Combining with Prasad's results for the trilinear GL_2 model ([P90]) and the fact that $m(\pi)=m(\tau)$, we prove Theorem 1.2.2.

If $\bar{Q}$ is of Type II, by Corollary 6.6.3, $m(\pi)=1$. Hence it is enough to prove the following proposition.

Proposition 13.3.1. *If $\bar{Q}$ is of Type II, we have $\epsilon(1/2,\pi,\wedge^3)=1$.*

Proof. Since $\bar{Q}$ is of Type II, it is contained in some Type II maximal parabolic subgroups. There are only two Type II maximal parabolic subgroups: type $(5,1)$ and type $(3,3)$.

If $\bar{Q}$ is contained in the parabolic subgroup $Q_{5,1}$ of type $(5,1)$, then there exists a tempered representation $\sigma=\sigma_1\otimes\sigma_2$ of $\mathrm{GL}_5(F)\times\mathrm{GL}_1(F)$ such that $\pi=I_{Q_{5,1}}^G(\sigma)$. Let ϕ_i be the Langlands parameter of σ_i for $i=1,2$. Then $\phi=\phi_1\oplus\phi_2$ is the Langlands parameter for π. Hence we have

$$\wedge^3(\phi) \;=\; \wedge^3(\phi_1\oplus\phi_2)=\wedge^3(\phi_1)\oplus(\wedge^2(\phi_1)\otimes\phi_2).$$

Since the central character of π is trivial, $\det(\phi)=\det(\phi_1)\otimes\det(\phi_2)=1$. Therefore $(\wedge^3(\phi_1))^\vee=\wedge^2(\phi_1)\otimes\det(\phi_1)^{-1}=\wedge^2(\phi_1)\otimes\det(\phi_2)=\wedge^2(\phi_1)\otimes\phi_2$. This implies

$$\epsilon(1/2,\pi,\wedge^3)=\det(\wedge^3(\phi_1))(-1)=(\det(\phi_1))^6(-1)=1.$$

If $\bar{Q}$ is contained in the parabolic subgroup $Q_{3,3}$ of type $(3,3)$, then there exists a tempered representation $\sigma=\sigma_1\otimes\sigma_2$ of $\mathrm{GL}_3(F)\times\mathrm{GL}_3(F)$ such that $\pi=I_{Q_{3,3}}^G(\sigma)$. Let ϕ_i be the Langlands parameter of σ_i for $i=1,2$. Then $\phi=\phi_1\oplus\phi_2$ is the Langlands parameter for π. Hence we have

$$\begin{aligned}\wedge^3(\phi) &= \wedge^3(\phi_1\oplus\phi_2)\\ &= (\wedge^2(\phi_1)\otimes\phi_2)\oplus(\phi_1\otimes\wedge^2(\phi_2))\oplus\det(\phi_1)\oplus\det(\phi_2).\end{aligned}$$

Since the central character of π is trivial, $\det(\phi)=\det(\phi_1)\otimes\det(\phi_2)=1$. Therefore $(\wedge^2(\phi_1)\otimes\phi_2)^\vee=(\phi_1\otimes\det(\phi_1)^{-1})\otimes(\wedge^2(\phi_2)\otimes\det(\phi_2)^{-1})=\phi_1\otimes\wedge^2(\phi_2)$ and $(\det(\phi_1))^\vee=$

$\det(\phi_2)$. This implies

$$\begin{aligned}
\epsilon(1/2, \pi, \wedge^3) &= \det(\wedge^2(\phi_1) \otimes \phi_2)(-1) \times \det(\phi_1)(-1) \\
&= \det(\wedge^2(\phi_1))^3(-1) \times \det(\phi_2)^3(-1) \times \det(\phi_1)(-1) \\
&= (\det(\phi_1)^2(-1))^3 \times (\det(\phi_2)(-1))^3 \times \det(\phi_1)(-1) = 1.
\end{aligned}$$

This finishes the proof of the proposition and hence the proof of Theorem 1.2.2. $\square$

Chapter 14

The Generic Case

In this Chapter, by applying the open orbit method, we prove some partial results for the general generic representations when F is archimedean. In Section 14.1, we consider the complex case and we will prove Theorem 1.2.3. In Section 14.2, we consider the real case and we will prove Theorem 1.2.4. Finally in Section 14.3, we will talk about how to remove the extra assumptions on Theorem 1.2.3(2) and Theorem 1.2.4(1) based on the results on the holomorphic continuation of the generalized Jacquet integral due to Raul Gomez in [G].

14.1 The Case When $F = \mathbb{C}$

In this section we assume that $F = \mathbb{C}$. By the same computation as in Section 7.1, we know that the epsilon factor is always 1. Hence we only need to prove that $m(\pi) = 1$. By the strong multiplicity one theorem, we only need to show that $m(\pi) \neq 0$.

We first consider the first part of Theorem 1.2.3. In other words, with the same notation as in Chapter 1, we assume that $\bar{P} \subset Q$. Then there are four possibilities for Q: type (6), type $(4,2)$, type $(2,4)$ or type $(2,2,2)$. The idea is to first reduce our problem to the reduced model $(L, R \cap Q)$ by the open orbit method, then reduce it to the tempered case which has been considered in Chapter 7.

If $Q = G$ **is of type** (6), by twisting π by some characters, we can assume that π is tempered. Note that twisting by characters will not change the multiplicities. Then by applying the result in Chapter 7, we know that $m(\pi) = 1$ and this proves Theorem

1.2.3.

If Q is of type $(4,2)$, then $L = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ and $R_Q = R \cap Q$ is of the form

$$R_Q = HU_{0,Q}$$

where

$$U_{0,Q}(F) = \{u = u(X) := \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid X \in M_2(F)\}.$$

The restriction of the character ξ on $U_{0,Q}(F)$ is just $\xi(u(X)) = \psi(\mathrm{tr}(X))$ and the character ω on H is defined as usual. By the definition of Q, π is of the form $I_Q^G(\tau_1|\ |^{t_1} \otimes \tau_2|\ |^{t_2})$ where τ_1, τ_2 are tempered and $t_1 < t_2$. Hence any element $f \in \pi$ is a smooth function $f : G(F) \to \tau = \tau_1|\ |^{t_1} \otimes \tau_2|\ |^{t_2}$ such that

$$f(lug) = \delta_Q(l)^{1/2}\tau(l)f(g) \tag{14.1}$$

for all $l \in L(F), u \in U_Q(F)$ and $g \in G(F)$. Here we use the letters π, σ, τ to denote both the representations and the underlying vector spaces. Let $\bar{Q} = LU_{\bar{Q}}$ be the opposite parabolic subgroup of Q. It is easy to see that $U_{\bar{Q}} \subset U$ and $U = U_{\bar{Q}}U_{0,Q}$. For any $f \in \pi$, define

$$J_Q(f) = \int_{U_{\bar{Q}}(F)} f(u)\xi^{-1}(u)du. \tag{14.2}$$

By Proposition 2.6.1 together with the assumption that $t_1 < t_2$, the integral above is absolutely convergent.

Proposition 14.1.1. *1. For all $f \in \pi, u \in U_{\bar{Q}}(F)$ and $l \in R_Q(F)$, we have*

$$J_Q(\pi(u)f) = \xi(u)J(f) \tag{14.3}$$

and

$$J_Q(\pi(l)f) = \tau(l)J(f). \tag{14.4}$$

2. The function

$$J_Q : \pi \to \tau,\ f \to J_Q(f)$$

is surjective.

Proof. Part (1) follows from (14.1) and changing variables in the integral (14.2). For part (2), fix a function $\varphi \in C_c^{\infty}(U_{\bar{Q}}(F))$ such that $\int_{U_{\bar{Q}}(F)} \varphi(u)\xi^{-1}(u)du = 1$. For any $v \in \tau$, since $Q(F)U_{\bar{Q}}(F)$ is open in $G(F)$, the function

$$f(g) = \begin{cases} \delta_Q(l)^{1/2}\tau(l)\varphi(u)v & \text{if} \quad g = u'lu \; with \; l \in L(F), u \in U_{\bar{Q}}(F), u' \in U_Q(F); \\ 0 & \text{else} \end{cases}$$

belongs to π. Then we have

$$J_Q(f) = \int_{U_{\bar{Q}}(F)} f(u)\xi^{-1}(u)du = \int_{U_{\bar{Q}}(F)} \varphi(u)\xi^{-1}(u)vdu = v.$$

This proves (2). □

We consider the Hom space $Hom_{R_Q(F)}(\tau, (\omega \otimes \xi)|_{R_Q(F)})$ and let $m(\tau)$ be the dimension of this space. The following proposition tells us the relation between $m(\pi)$ and $m(\tau)$.

Proposition 14.1.2.

$$m(\tau) \neq 0 \Rightarrow m(\pi) \neq 0.$$

Proof. If $m(\tau) \neq 0$, choose $0 \neq l_0 \in Hom_{R_Q(F)}(\tau, (\omega \otimes \xi)|_{R_Q(F)})$. Define an operator l on π to be

$$l(f) = l_0(J_Q(f)).$$

Since $l_0 \neq 0$ and J_Q is surjective, we have $l \neq 0$. Hence we only need to show that $l \in Hom_{R(F)}(\pi, \omega \otimes \xi)$.

For $h \in R(F)$, we can write $h = h_1 u_1$ with $h_1 \in R_Q(F)$ and $u_1 \in U_{\bar{Q}}(F)$. By (14.3) and (14.4), we have

$$\begin{aligned} l(\pi(h)f) &= l_0(J_Q(\pi(h_1u_1)f)) = l_0(\tau(h_1)J_Q(\pi(u_1)f)) \\ &= \omega \otimes \xi(h_1)l_0(J_Q(\pi(u_1)f)) = \omega \otimes \xi(h_1)l_0(\xi(u_1)J_Q(f)) \\ &= \omega \otimes \xi(h)l_0(J_Q(f)) = \omega \otimes \xi(h)l(f). \end{aligned}$$

This implies $l \in Hom_{R(F)}(\pi, \omega \otimes \xi)$ and finishes the proof of the Proposition. □

By the proposition above, we only need to show that $m(\tau) \neq 0$. It is easy to see that the multiplicity $m(\tau)$ is invariant under the unramified twist, hence we may assume that

τ is tempered (note that originally τ is of the form $\tau_1|\ |^{t_1} \otimes \tau_2|\ |^{t_2}$ with τ_1 and τ_2 being tempered). Then by applying the argument in Chapter 7 to the middle model case, we can show that the multiplicity $m(\tau)$ is always nonzero for all tempered representations τ. This proves Theorem 1.2.3.

If Q **is of type** $(2,4)$, the argument is the same as the $(4,2)$ case, we will skip it here.

If Q **is of type** $(2,2,2)$, the argument is still similar to the $(4,2)$ case: we first reduce to the trilinear GL_2 model case by the open orbit method. Then after twisting by some characters we only need to consider the tempered case. Finally, by applying the argument in Chapter 7 to the trilinear GL_2 model case, we can show that the multiplicity is nonzero and this proves Theorem 1.2.3. We will skip the details here.

Now the proof of Theorem 1.2.3(1) is complete.

Then we consider the second part of Theorem 1.2.3. As in Chapter 1, we assume that $\pi = I_B^G(\otimes_{i=1}^6 \chi_i)$ where B is the lower Borel subgroup, $\chi_i = \sigma_i|\ |^{s_i}$, σ_i are unitary characters, and s_i are real numbers with $s_1 \leq s_2 \leq \cdots \leq s_6$. By the assumption $Q \subset \bar{P}$, we have $s_2 < s_3$ and $s_4 < s_5$. Also as in Section 1, we write $\pi = I_{\bar{P}}^G(\pi_0)$ with $\pi_0 = \pi_1 \otimes \pi_2 \otimes \pi_3$ and π_i be the parabolic induction of $\chi_{2i-1} \otimes \chi_{2i}$. Then π consists of smooth functions $f \to \pi_0$ such that

$$f(mug) = \delta_{\bar{P}}(m)^{1/2}\pi_0(m)f(g) \tag{14.5}$$

for all $m \in M(F), u \in \bar{U}(F)$ and $g \in G(F)$. We still want to apply the open orbit method. For $f \in \pi$, define

$$J(f) = \int_{U(F)} f(ug)\xi^{-1}(u)du. \tag{14.6}$$

By Proposition 2.6.1 together with the assumption on the exponents s_i, the integral above is absolutely convergent. Similarly as in the previous situation, we can show that

$$m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0. \tag{14.7}$$

Here $m(\pi_0)$ is the multiplicity for the trilinear GL_2 model. In fact, for $0 \neq l_0 \in Hom_{H_0(F)}(\pi_0, \omega)$. By a similar argument as in Proposition 14.1.2, we know that

$$l(f) := l_0(J(f))$$

is a nonzero element in $Hom_{R(F)}(\pi, \omega \otimes \xi)$. This proves (14.7). Now by our assumption on π_0 together with the work by Loke for the trilinear GL_2 model in [L01], we know that $m(\pi_0) \neq 0$. This implies we have $m(\pi) \neq 0$ and finishes the proof of Theorem 1.2.3.

Remark 14.1.3. *The assumption $Q \subset \bar{P}$ is only used to make the generalized Jacquet integral $J(f)$ to be absolutely convergent. Hence in general, if one can prove the holomorphic continuation of the generalized Jacquet integral $J(f)$, then the assumption $Q \subset \bar{P}$ in Theorem 1.2.3(2) can be removed. This will be discussed in Section 14.3.*

14.2 The Case When $F = \mathbb{R}$

In this section by applying the open orbit method to the case when $F = \mathbb{R}$, we prove Theorem 1.2.4. Let π be an irreducible generic representation of $G(F)$ with central character χ^2. With the same notation as in Chapter 1, there is a parabolic subgroup $Q = LU_Q$ containing the lower Borel subgroup and an essential tempered representation $\tau = \otimes_{i=1}^{k} \tau_i |\ |^{s_i}$ of $L(F)$ with τ_i tempered, $s_i \in \mathbb{R}$ and $s_1 < s_2 < \cdots < s_k$ such that $\pi = I_Q^G(\tau)$.

We first consider the case when $\pi_D = 0$. Then by our assumptions in Theorem 1.2.4, Q is nice. If $Q \subset \bar{P}$, let $\pi_0 = I_{Q\cap M}^{M}(\tau)$, it is a generic representation of $M(F)$ and we have $\pi = I_{\bar{P}}^{G}(\pi_0)$. By the same argument as in previous section, we can show that

$$m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0 \tag{14.8}$$

where $m(\pi_0)$ is the multiplicity of the trilinear GL_2 model. Since $\pi_D = 0$, the Jacquet-Langlands correspondence of π_0 from $M(F) = (\mathrm{GL}_2(F))^3$ to $(\mathrm{GL}_1(D))^3$ is zero. By applying the result for the trilinear GL_2 model in [P90] and [L01], we have $m(\pi_0) = 1$. Combining with (14.8), we know $m(\pi) \neq 0$. Hence $m(\pi) = 1$ since we already know $m(\pi) \leq 1$. Therefore

$$m(\pi) + m(\pi_D) = m(\pi) = 1.$$

This proves Conjecture 1.1.2. For Conjecture 1.1.3, we only need to show that when $\pi_D = 0$, the epsilon factor $\epsilon(1/2, \pi, \wedge^3)$ is always 1. Since $\pi_D = 0$, by the local Jacquet-Langlands correspondence in [DKV84], π_0 is not an essential discrete series (i.e. discrete series twisted by characters), hence at least one of the π_i $(i = 1, 2, 3)$ is a principal series.

Therefore we can find a generic representation $\sigma = \sigma_1 \otimes \sigma_2$ of $\mathrm{GL}_5(F) \times \mathrm{GL}_1(F)$ such that π is the parabolic induction of σ. Then by the same argument as in Chapter 7, we can show that

$$\epsilon(1/2, \pi, \wedge^3) = 1.$$

This finishes the proof of Conjecture 1.1.3.

If $Q \subset \bar{P}$, there are only four possibilities for Q: type $(6), (4,2), (2,4)$ and $(2,2,2)$. If Q is type (6), by twisting π by some characters we can assume that π is tempered, then both Conjecture 1.1.2 and Conjecture 1.1.3 are proved in Chapter 7. If Q is type $(4,2)$ or $(2,4)$, by the same argument as in previous section, we can reduce to the middle model case by the open orbit method. Then by twisting some characters, we only need to consider the tempered case which has already been proved in Chapter 7. If Q is type $(2,2,2)$, the argument is similar except replacing the middle model by the trilinear GL_2 model.

Now the proof of Theorem 1.2.4(1) is complete.

Then we consider the case when $\pi_D \neq 0$. As a result, $\pi = I_{\bar{P}}^{G}(\pi_0)$ is the parabolic induction of some essential discrete series $\pi_0 = \pi_1|\ |^{s_1} \otimes \pi_2|\ |^{s_2} \otimes \pi_3|\ |^{s_3}$ of $M(F)$ where π_i are discrete series of $\mathrm{GL}_2(F)$ and s_i are real numbers. As usual, we assume that $s_1 \leq s_2 \leq s_3$. On the mean time, π_D is of the form $I_{\bar{P}_D}^{G_D}(\pi_{0,D})$ where $\pi_{0,D} = \pi_{1,D}|\ |^{s_1} \otimes \pi_{2,D}|\ |^{s_2} \otimes \pi_{3,D}|\ |^{s_3}$ is the Jacquet-Langlands correspondence of π_0 from $M(F)$ to $M_D(F)$. Let $m(\pi_0)$ (resp. $m(\pi_{0,D})$) be the multiplicity of the trilinear $\mathrm{GL}_2(F)$ (resp. $\mathrm{GL}_1(D)$) model.

Proposition 14.2.1. *With the notations above, in order to prove Theorem 1.2.4(2), it is enough to show that*

$$m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0;\ m(\pi_{0,D}) \neq 0 \Rightarrow m(\pi_D) \neq 0. \tag{14.9}$$

Proof. By Prasad's result for the trilinear GL_2 model, we have

$$m(\pi_0) + m(\pi_{0,D}) = 1. \tag{14.10}$$

Moreover, if we assume that the central character of π_0 is trivial on $Z_H(F)$, we have

$$m(\pi_0) = 1 \iff \epsilon(1/2, \pi_0) = 1;\ m(\pi) = 0 \iff \epsilon(1/2, \pi_0) = -1. \tag{14.11}$$

Combining (14.9) and (14.10), we have $m(\pi)+m(\pi_D)\geq 1$, this proves the first part of Theorem 1.2.4(2). For the second part, assume that the central character of π is trivial, as proved in Section 7.3, we have

$$\epsilon(1/2,\pi,\wedge^3)=\epsilon(1/2,\pi_0). \tag{14.12}$$

Now if $\epsilon(1/2,\pi,\wedge^3)=1$, by (14.12), we have $\epsilon(1/2,\pi_0)=1$. Combining with (14.11), we have $m(\pi_0)=1$, therefore $m(\pi)=1$ by (14.9). On the other hand, if $m(\pi)=0$, by (14.9), we have $m(\pi_0)=0$. Combining with (14.11), we have $\epsilon(1/2,\pi_0)=-1$, therefore $\epsilon(1/2,\pi,\wedge^3)=-1$ by (14.12). This finishes the proof of Theorem 1.2.4(2). $\square$

By the proposition above, it is enough to prove (14.9). If $s_1=s_2=s_3$, by twisting π by some characters, we may assume that π is tempered (note that the multiplicities for both the Ginzburg-Rallis model the the trilinear GL_2 model are invariant under twisting by characters). Then the relation (14.9) has already been proved in Corollary 6.6.2. In fact, by Corollary 6.6.2, we even have $m(\pi)=m(\pi_0)$ and $m(\pi_D)=m(\pi_{0,D})$.

If $s_1<s_2=s_3$, let $\pi_{2,3}$ be the parabolic induction of $\pi_2\otimes\pi_3$, it is a tempered representation of $\mathrm{GL}_4(F)$. We also know that π will be the parabolic induction of $\pi'=\pi_1|\ |^{s_1}\otimes\pi_{2,3}|\ |^{s_2}$. Let $m(\pi')$ be the multiplicity for the middle model. By applying the open orbit method as in the previous section, we have

$$m(\pi')\neq 0\Rightarrow m(\pi)\neq 0.$$

Hence in order to prove $m(\pi_0)\neq 0\Rightarrow m(\pi)\neq 0$, it is enough to show that $m(\pi_0)\neq 0\Rightarrow m(\pi')\neq 0$. Again by twisting π' by some characters, we may assume that π' is tempered. Then by applying Corollary 6.6.2 again, we have $m(\pi_0)=m(\pi')$ which implies $m(\pi_0)\neq 0\Rightarrow m(\pi)\neq 0$. The proof of the quaternion version is similar. This proves (14.9).

If $s_1=s_2<s_3$, the argument is the same as the case above, we will skip it here.

If $s_1<s_2<s_3$, (14.9) follows directly from the open orbit method as in the previous section.

Now the proof of Theorem 1.2.4(2) is complete.

14.3 Holomorphic Continuation of the Generalized Jacquet Integrals

In the previous sections, we have already seen that the extra conditions of Q in Theorem 1.2.3(2) and Theorem 1.2.4(1) can be removed if the generalized Jacquet integral $J(f)$ defined in (14.6) has holomorphic continuation. In this section, we are going to remove the condition on Q based on the following hypothesis.

Hypothesis: The generalized Jacquet integrals have holomorphic continuation for all parabolic subgroups whose unipotent radical is abelian.

The Hypothesis has been proved by Gomez and Wallach in [GW12] for the case when the stabilizer of the unipotent character is compact, and proved by Gomez in [G] for the general case. The second paper is still in preparation, this is why we write it as a hypothesis.

Let $F=\mathbb{R}$ or $\mathbb{C}$, π be a generic representation of $\mathrm{GL}_6(F)$ of the form $\pi=I_{\bar{P}}^{G}(\pi_0)$ for some generic representation π_0 of $M(F)=(\mathrm{GL}_2(F))^3$. By the discussion in Section 14.1 and 14.2, we know that in order to prove Theorem 1.2.3(2) and Theorem 1.2.4(1) for π, it is enough to show that

$$m(\pi_0)\neq 0\Rightarrow m(\pi)\neq 0. \tag{14.13}$$

where $m(\pi_0)$ is the multiplicity for the trilinear GL_2 model.

Let $Q_{4,2}=L_{4,2}U_{4,2}$ be the parabolic subgroup of $\mathrm{GL}_6(F)$ containing $\bar{P}$ of type $(4,2)$, and let $\pi_1=I_{\bar{P}\cap L_{4,2}}^{L_{4,2}}(\pi_0)$. Then in order to prove (14.13), it is enough to show that

$$m(\pi_0)\neq 0\Rightarrow m(\pi_1)\neq 0,\ m(\pi_1)\neq 0\Rightarrow m(\pi)\neq 0 \tag{14.14}$$

where $m(\pi_1)$ is the multiplicity for the middle model. Note that the unipotent radicals of $Q_{4,2}$ and $\bar{P}\cap L_{4,2}$ are all abelian. Therefore by the hypothesis, the generalized Jacquet integrals associated to $Q_{4,2}$ and $\bar{P}\cap L_{4,2}$ have holomorphic continuation. This allows us to apply the open orbit method as in the previous sections, which give the relations in (14.14). This proves (14.13), and finishes the proof of Theorem 1.2.3(2) and Theorem 1.2.4(1) without the assumptions on Q.

References

[Ar75] J. Arthur, *A theorem on the Schwartz space of a reductive Lie group.* Proc. Nat. Acad. Sci. U.S.A. 72 (1975), no. 12, 4718-4719

[Ar81] J. Arthur, *The trace formula in invariant form.* Ann. of Math. (2) 114 (1981), 1–74

[Ar87] J. Arthur, *The characters of supercuspidal representations as weighted orbital integral.* Proc. Indian Acad. Sci. 97 (1987), 3–19.

[Ar89] J. Arthur, *Intertwining operators and residues I. Weighted characters,* J. Funct. Analysis 84 (1989), 19-84.

[Ar91] J. Arthur, *A local trace forumula.* Publ. Math. Inst. Hautes Études Sci. 73(1991), 5–96

[Ar93] J. Arthur, *On elliptic tempered characters.* Acta Math. 171 (1993), 73–138

[B12] R. Beuzart-Plessis, *La conjecture locale de Gross-Prasad pour les reprsentations tempérées des groupes unitaires.* Prepublication, 2012.

[B15] R. Beuzart-Plessis, *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the archimedean case.* Preprint, 2015.

[Ber88] J. Bernstein, *On the support of Plancherel measure.* Jour. of Geom. and Physics 5, No.4 1988.

[BDK] J. Bernstein, P. Deligne, D. Kazhdan, *Trace Paley-Wiener theorem for reductive p-adic groups.* J. d'Analyse Math. 47 (1986), 441-472.

[BK14] J. Bernstein, B. Krotz, *Smooth Fréchet globalizations of Harish-Chandra modules.* Israel J. Math. 199 (2014), no. 1, 45-111.

[DKV84] P. Deligne, D. Kazhdan, M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques.* Représentations des groupes réductifs sur un corps local, Travaux en Cours, Paris(1984): Hermann, pp. 33–117

[G] R. Gomez, *Bessel models for general admissible induced representations: the noncompact stabilizer case.* In preparation.

[GGP12] Wee Teck Gan, B. Gross; D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups.* Sur les conjectures de Gross et Prasad. I. Astérisque No. 346 (2012), 1–109.

[GR00] D. Ginzburg, S. Rallis, *The exterior cube L-function for GL(6).* Compositio Math. 123(2000), no. 3, 243–272

[GP92] B. Gross, D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}*.* Can. J. Math. Vol. 44 (1992), 974–1002

[GP94] B. Gross, D. Prasad, *On irreducible representations of* $SO_{2n+1} \times SO_{2m}$*.* . Canad. J. Math. Vol. 46 (1994), no. 5, 930–950.

[GW12] R. Gomez, N. Wallach, *Bessel models for general admissible induced representations: the compact stabilizer case.* Selecta Math. (N.S.) 18 (2012), no. 1, 1-26.

[HK04] M. Harris, S. Kudla, *On a conjecture of Jacquet. Contributions to automorphic forms, geometry, and number theory.* Johns Hopkins Univ. Press, 2004, 355–371

[J08] Dihua Jiang, *Residues of Eisenstein series and related problems.* Eisenstein Series and Applications, Progress in Math, 258, 187–204, 2008

[JS90] H. Jacquet, J. Shalika, *Rakin-Selberg convolutions: Archimedean theory.* In Festschrift in honor of I.I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I, Israel Math. Conf. Proc., vol. 2, Weizmann, 1990, 125-207.

[JSZ11] Dihua Jiang, Bingyong Sun, Chengbo Zhu, *Uniqueness of the Ginzburg-Rallis models: the archimedean case.* Trans. Amer. Math. Soc. 363 (2011), 2763–2802.

[K05] R. E. Kottwitz, *Harmonic analysis on reductive p-adic groups and Lie algebras.* in "Harmonic analysis, the trace formula, and Shimura varieties", 393522, Clay Math. Proc. 4, Amer. Math. Soc., Providence, RI, 2005

[KKSS] F. Knop, B. Krotz, E. Sayag, H. Schlichtkrull, *Simple compactifications and polar decomposition of homogeneous real spherical spaces.* prepublication 2014

[Knop95] Friedrich Knop, *On the set of orbits for a Borel subgroup.* Comment. Math. Helv., 70(2):285-309, 1995

[KR94] S. Kudla, S. Rallis, *A regularized Siegel-Weil formula: the first term identity.* Ann. Math. 140 (1994), 1–80.

[L01] Loke, Hung Yean *Trilinear forms of* $\mathfrak{gl}_2$*.* Pacific J. Math. 197 (2001), no. 1, 119C144.

[MW12] Colette Moeglin, J.-L. Waldspurger, *La conjecture locale de Gross-Prasad pour les groupes spciaux orthogonaux: le cas gnral.* in "Sur les conjectures de Gross et Prasad. II" Astérisque No. 347 (2012), 167-216

[N06] C. Nien, *Models of representations of general linear groups over p-adic fields.* PHD Thesis, University of Minnesota, 2006.

[P90] D. Prasad, *Trilinear forms for representations of GL(2) and local (epsilon)-factors.* Compositio Math. 75 (1990), 1–46

[P92] D. Prasad, *Invariant forms for representations of GL(2) over a local field.* American J. of Maths, vol. 114, 1317-1363 (1992).

[Rod81] F. Rodier, *Modèle de Whittaker et caractères de représentations.* Noncommutative harmonic analysis, Lecture Notes in Mathematics, vol.466, eds J. Carmona, J. Dixmier and M. Vergne(Springer, Berlin, 1981), 151–171.

[S12] Y. Sakellaridis, *Spherical varieties and integral representations of L-functions.* Algebra and Number Theory, Vol. 6 (2012), No. 4, 611-667.

[S13] Yiannis Sakellaridis, *Spherical functions on spherical varieties.* Amer. J. Math., 135(5):1291C1381, 2013.

[SV] Y. Sakellaridis, A. Venkatesh, *Periods and harmonic analysis on spherical varieties.* Prepublication 2012

[W03] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques(d'après Harish-Chandra)*, J. Inst. Math. Jussieu 2 (2003), no.2, 235-333

[W10] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad.* Compos. Math. 146(2010), no.5, 1180–1290.

[W12] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2e partie : extension aux représentations tempérées.* in "Sur les conjectures de Gross et Prasad. I" Astérisque No. 346 (2012), 171–312

[Wan15] Chen Wan, *A Local Relative Trace Formula for the Ginzburg-Rallis Model: the Geometric Side.* 95 pages, accepted by the Memoirs of the AMS, 2015.

[Wan16] Chen Wan, *Multiplicity One Theorem for the Ginzburg-Rallis Model: the tempered case.* Submitted, 2016. 74 pages

[Wan16b] Chen Wan, *The local Ginzburg-Rallis model over complex field.* Submitted, 2016. 17 pages

Appendix A

The Cartan Decomposition

A.0.1 The problem

In this Appendix, we are going to prove the weak Cartan decomposition for the trilinear GL_2 model (as in Proposition 4.2.3). Let F be a p-adic field, $\mathcal{O}_F$ be the ring of integers, ϖ_F be the uniformizer, $|\ | = |\ |_F$, and let $\mathbb{F}_q$ be the residue field with $q = p^n$. Let $G = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, $H = \mathrm{GL}_2(F)$ diagonally embedded into G, $K' = \mathrm{GL}_2(\mathcal{O}_F) \cup \mathrm{GL}_2(\mathcal{O}_F) \begin{pmatrix} 1 & 0 \\ 0 & \varpi_F \end{pmatrix}$, $K_0 = \mathrm{GL}_2(\mathcal{O}_F) \times \mathrm{GL}_2(\mathcal{O}_F) \times \mathrm{GL}_2(\mathcal{O}_F)$ be the maximal compact subgroup of G, $K = K_0(K' \times K' \times K')K_0$ be a compact subset of G with $K = K_0 K K_0$, and let

$$A^+ = \{(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} a_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, a_2, a_3) | a_1, a_2 \in A_0^-,\ a_3 \in A_0^+\}$$

where $A_0^+ = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in F^\times,\ |a| \geq |b|\}$ and $A_0^- = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in F^\times,\ |a| \leq |b|\}$. Our goal is to show that

$$G = HA^+K. \tag{A.1}$$

We first do some reductions. For $(g_1, g_2, g_3) \in G$, by timing some elements on K^{-1} on the right and by timing some elements in the center (which is contained in A^+), may assume that $\det(g_1) = \det(g_2) = \det(g_3) = 1$. Then by timing $(g_1^{-1}, g_1^{-1}, g_1^{-1}) \in H$ on the left, we only need to consider elements of the form $(1, g, g')$. Applying the Cartan decomposition $\mathrm{GL}_2(F) = \mathrm{GL}_2(\mathcal{O}) A_0^+ \mathrm{GL}_2(\mathcal{O})$ to g and g', then by absorbing the right $\mathrm{GL}_2(\mathcal{O}_F)$

part by elements in K_0, we only need to consider elements of the form $(1, ka, k'a')$ with $k, k' \in \mathrm{GL}_2(\mathcal{O}_F)$ and $a, a' \in A_0^+$. Then by timing $(a^{-1}k^{-1}, a^{-1}k^{-1}, a^{-1}k^{-1}) \in H$ on the left, and absorbing k^{-1} by elements in K_0, we only need to consider elements of the form $(a, 1, g)$ with $a \in A_0^-$ and $g \in \mathrm{GL}_2(F)$. Applying the Iwasawa decomposition to g, we may assume that g is upper triangular. Therefore we only need to consider elements of the form

$$(a, a', b)$$

where $a \in A_0^-$ with $\det(a) = 1$, $a' = I_2$, and b is upper triangular with $\det(b) = 1$. Then by timing $(u, u, u) \in H$ on the left with $u = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and absorbing the u in the second coordinate by elements in K_0, we only need to consider elements of the form

$$(ua, a', b) \tag{A.2}$$

where $a \in A_0^-$ with $\det(a) = 1$, $a' = I_2$, and b is upper triangular with $\det(b) = 1$. By the discussion above, in order to prove (A.1), it is enough to prove the following proposition.

Proposition A.0.1. *For all elements $g = (ua, a', b)$ of the form* (A.2)*, there exist $h \in H(F)$, $t \in A^+$ and $k \in K_0$ such that*

$$g = htk.$$

A.0.2 The case when b is diagonal

In this section, we prove Proposition A.0.1 for the case when b is a diagonal matrix. We let $a = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$ with $|x| \geq 1$. By our assumption, $b = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ or $\begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}$ with $|y| \geq 1$.

Case 1: If $b = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$, let

$$h = (I_2, I_2, I_2),\ t = (uau^{-1}, I_2, b) \in A^+,\ k = (u, I_2, I_2).$$

Then we have

$$g = htk.$$

Case 2: If $b = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}$, let

$$h = (\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}),\ t = (uau^{-1}, I_2, b^{-1})$$

and

$$k = (u\begin{pmatrix} 1 & 0 \\ -\frac{1}{x^2} & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{y^2} & 1 \\ -1 & 0 \end{pmatrix}).$$

Then we have

$$g = htk.$$

This proves Proposition A.0.1 when b is a diagonal matrix.

A.0.3 The general situation

In this section, we prove Proposition A.0.1 for the general case (i.e. b is a upper triangular matrix). We still let $a = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$ with $|x| \geq 1$. The proof breaks into four cases.

Case 1: If $b = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $|a| \geq |b|$, then $b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$ with $\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$. Then by timing some elements in K_0, we reduce to the case when b is a diagonal matrix, which has been considered in the previous section.

Case 2: If $b = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ with $|y| \geq 1$. If $|t| \leq |y|^2$, we are back to case 1. So we may assume that $|t| > |y|^2 \geq 1$. Let

$$h = (\begin{pmatrix} 1-t^{-1} & 0 \\ t^{-1} & 1 \end{pmatrix}, \begin{pmatrix} 1-t^{-1} & 0 \\ t^{-1} & 1 \end{pmatrix}, \begin{pmatrix} 1-t^{-1} & 0 \\ t^{-1} & 1 \end{pmatrix}),\ t = (uau^{-1}, I_2, \begin{pmatrix} \frac{t}{y} & 0 \\ 0 & \frac{y}{t} \end{pmatrix})$$

and

$$k = (u\begin{pmatrix} 1 & 0 \\ \frac{1}{x^2 t} & 1-t^{-1} \end{pmatrix}^{-1}, \begin{pmatrix} 1-t^{-1} & 0 \\ t^{-1} & 1 \end{pmatrix}^{-1}, \begin{pmatrix} -\frac{1}{y^2} & -1 \\ 1 & \frac{y^2}{t} \end{pmatrix}^{-1}).$$

Then we have

$$g = htk.$$

Case 3: If $b = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}$ with $|y| \geq 1$ and $|t| > 1$. Let

$$h = (\begin{pmatrix} \frac{t}{t+1} & 0 \\ \frac{1}{t+1} & 1 \end{pmatrix}, \begin{pmatrix} \frac{t}{t+1} & 0 \\ \frac{1}{t+1} & 1 \end{pmatrix}, \begin{pmatrix} \frac{t}{t+1} & 0 \\ \frac{1}{t+1} & 1 \end{pmatrix}),\ t = (uau^{-1}, I_2, \begin{pmatrix} yt & 0 \\ 0 & \frac{1}{yt} \end{pmatrix})$$

and

$$k = (u \begin{pmatrix} 1 & 0 \\ \frac{1}{x^2(t+1)} & \frac{t}{t+1} \end{pmatrix}^{-1}, \begin{pmatrix} \frac{t}{t+1} & 0 \\ \frac{1}{t+1} & 1 \end{pmatrix}^{-1}, \begin{pmatrix} 0 & -1 \\ \frac{t}{t+1} & \frac{1}{ty^2} \end{pmatrix}^{-1}).$$

Then we have

$$g = htk.$$

Case 4: If $b = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}$ with $|y|, |t| \geq 1$. If $|t| \geq |y|^2$, we are back to case 1. So we may assume that $1 \leq |t| < |y|^2$. There are two subcases.

Case 4(a): If $|t| \geq |x|^2$. We time g by $(\begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix})$ on the left. Note that $a^{-1}u^{-1} \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix} ua = \begin{pmatrix} 1 & x^2t^{-1} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$. Hence by modulo an element in K_0, we may assume that $b = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}$ is a diagonal matrix, which has been considered in the previous section.

Case 4(b): If $1 \leq |t| < |x|^2$. We have three subcases.

Case 4(b)(i): If $|t+1| \geq 1$. Let

$$h = (\begin{pmatrix} \frac{1}{t} & 1 + \frac{1}{t(t+1)} \\ 1 & \frac{1}{t+1} \end{pmatrix}, \begin{pmatrix} \frac{1}{t} & 1 + \frac{1}{t(t+1)} \\ 1 & \frac{1}{t+1} \end{pmatrix}, \begin{pmatrix} \frac{1}{t} & 1 + \frac{1}{t(t+1)} \\ 1 & \frac{1}{t+1} \end{pmatrix}),\ t = (uau^{-1}, I_2, \begin{pmatrix} y & 0 \\ 0 & \frac{1}{y} \end{pmatrix})$$

and

$$k = (u \begin{pmatrix} \frac{t+1}{t} & 0 \\ \frac{1}{x^2} & -\frac{t}{t+1} \end{pmatrix}^{-1}, \begin{pmatrix} \frac{1}{t} & 1 + \frac{1}{t(t+1)} \\ 1 & \frac{1}{t+1} \end{pmatrix}^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{(t+1)y^2} \end{pmatrix}^{-1}).$$

Then we have

$$g = htk.$$

Case 4(b)(ii): If $t = -1$. We time g by (u^{-1}, u^{-1}, u^{-1}) on the left and absorb the second u^{-1} by some elements in K_0. As a result, we may assume that

$g = (\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}, I_2, \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix})$ with $|x|, |y| \geq 1$. If $|y| \geq |x|$, let

$$h = (\begin{pmatrix} 0 & x^{-1} \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & x^{-1} \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & x^{-1} \\ x & 0 \end{pmatrix}),\ t = (I_2, \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x^{-1}y & 0 \\ 0 & xy^{-1} \end{pmatrix})$$

and

$$k = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

If $|y| < |x|$, let

$$h = (\begin{pmatrix} y^{-1} & y^{-1} \\ -y & 0 \end{pmatrix}, \begin{pmatrix} y^{-1} & y^{-1} \\ -y & 0 \end{pmatrix}, \begin{pmatrix} y^{-1} & y^{-1} \\ -y & 0 \end{pmatrix}),\ t = (u \begin{pmatrix} x^{-1}y & 0 \\ 0 & xy^{-1} \end{pmatrix} u^{-1}, \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}, I_2)$$

and

$$k = (u \begin{pmatrix} 1 & 0 \\ x^{-2}y^2 & 1 \end{pmatrix}, \begin{pmatrix} y^{-2} & 1 \\ -1 & 0 \end{pmatrix}^{-1}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1}).$$

Then in both cases, we have

$$g = htk.$$

Case 4(b)(iii): If $|t+1| < 1$ with $t \neq -1$, then $|t| = 1$. If $|(t+1)y^2| \leq 1$, we have

$$\begin{aligned} b &= \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1}+1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & (t^{-1}+1)y^2 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

with $\begin{pmatrix} 1 & (t^{-1}+1)y^2 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$. Then up to modulo an element in K_0, we can eliminate $\begin{pmatrix} 1 & (t^{-1}+1)y^2 \\ 0 & 1 \end{pmatrix}$, and we have reduced to Case 4(b)(ii).

If $|(t+1)x^2| \leq 1$, we time g by $(\begin{pmatrix} 1 & -t^{-1}+1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -t^{-1}+1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -t^{-1}+1 \\ 0 & 1 \end{pmatrix}) \in H$ on the left, then modulo an element in K_0 to eliminate $a^{-1}u^{-1} \begin{pmatrix} 1 & t^{-1}+1 \\ 0 & 1 \end{pmatrix} ua =$

$\begin{pmatrix} 1 & (t^{-1}+1)x^2 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$ in the first coordinate and $\begin{pmatrix} 1 & t^{-1}+1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$ in the second coordinate, we have still reduced to Case 4(b)(ii).

Now the only case left is when $|(t+1)y^2|, |(t+1)x^2| > 1$. Let

$$h = (\begin{pmatrix} \frac{1}{t+1} & 0 \\ \frac{t}{t+1} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{t+1} & 0 \\ \frac{t}{t+1} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{t+1} & 0 \\ \frac{t}{t+1} & 1 \end{pmatrix}),$$

$$t = (u \begin{pmatrix} x^{-1} & 0 \\ 0 & x(t+1) \end{pmatrix} u^{-1}, \begin{pmatrix} t+1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{y(t+1)}{t} & 0 \\ 0 & -\frac{t}{y} \end{pmatrix})$$

and

$$k = (u \begin{pmatrix} 1 & 0 \\ -\frac{t}{x^2(t+1)} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -y^{-2}t \end{pmatrix}^{-1}).$$

Then we have

$$g = htk.$$

The proof of Proposition A.0.1 is finally complete.

Appendix B

The Absolutely convergence of $I(f)$

In this appendix, we prove Proposition 8.1.1. The proof goes exactly the same as Proposition 7.1.1 of [B15]. In the loc. cit., the author is dealing with the Gan-Gross-Prosad model case, but the proof of that Proposition worked for general cases except the following five results which are specified to the GGP model case: Lemma 6.5.1, Lemma 6.6.1, Proposition 6.4.1, Proposition 6.7.1 and Proposition 6.8.1 in the loc. cit. But we already proved the above five results for the Ginzburg-Rallis model in Chapter 4, see Lemma 4.1.1, Proposition 4.2.1, Proposition 4.2.3, Lemma 4.3.1, Lemma 4.3.3, Proposition 4.4.1 and Lemma 4.4.2. Therefore the argument in the loc. cit. can be applied to our case smoothly. We only include the proof here for completion.

We first prove (1): for all $d'>0$, we have

$$|I(f,x)| \ll q_{d'}(f)\int_{Z_R(F)\backslash R(F)} \Xi^G(x^{-1}hx)\sigma_0(x^{-1}hx)^{-d'}dh$$

for all $f\in \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})$ and $x\in R(F)\backslash G(F)$. Here for all $f\in \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})$,

$$q_d(f)=sup_{g\in G(F)}|f(g)|\Xi^G(g)^{-1}\sigma_0(g)^d.$$

Then by Proposition 4.4.1(5), if d' is large enough, there exists $d>0$ such that

$$\int_{Z_R(F)\backslash R(F)} \Xi^G(x^{-1}hx)\sigma_0(x^{-1}hx)^{-d'}dh \ll \Xi^{R\backslash G}(x)^2\sigma_{R\backslash G}(x)^d$$

for all $x \in R(F)\backslash G(F)$. This proves (1).

For (2), we use the same notations as in Chapter 4. In other word,

- $\bar{P}_0 = M_0\bar{U}_0$ is a good minimal parabolic subgroup of G_0, $A_0 = A_{M_0}$.
- $A_0^+ = \{a_0 \in A_0(F) \mid \mid \alpha(a_0) \mid \geq 1 \; for\; all\; \alpha \in \Psi(A_0, \bar{P}_0)\}$.
- $\bar{P}_{min} = \bar{P}_0\bar{U} = M_{min}\bar{U}_{min}$ is a good minimal parabolic subgroup of G, $A_{min} = A_{M_{min}} = A_0$.
- $A_{min}^+ = \{a \in A_{min}(F) \mid \mid \alpha(a) \mid \geq 1 \; for\; all\; \alpha \in \Psi(A_{min}, \bar{P}_{min})\}$.
- Δ is the set of simple roots of A_{min} in P_{min}, and $\Delta_P = \Delta \cap \Psi(A_{min}, P)$.

Again by the weak Cartan decomposition in Section 4.2, it is enough to prove the estimation of the proposition for $x = a \in A_0^+$. Moreover, we can fix an open compact subgroup $K \subset G(F)$, and we only need to prove the following statement:

(i) For all $d > 0$, there exists a continuous seminorm $\nu_{d,K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that
$$|I(f,a)| \ll \nu_{d,K}(f)\Xi^{R\backslash G}(a)^2\sigma_{R\backslash G}(a)^{-d}$$
for all $a \in A_0^+$ and $f \in \mathcal{C}_{scusp,K}(Z_G(F)\backslash G(F), \eta^{-1})$.

We set
$$A_{min}^{a+} = \{a \in A_0^+ \mid |\alpha(a)| \leq \sigma_{R\backslash G}(a) \; \forall \alpha \in \Delta_P\}.$$

We first prove the following statement:

(ii) For all $d > 0$, there exists a continuous seminorm $\nu_{d,K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that
$$|I(f,a)| \ll \nu_{d,K}(f)\Xi^{R\backslash G}(a)^2\sigma_{R\backslash G}(a)^{-d}$$
for all $a \in A_0^+\backslash A_{min}^{a+}$ and $f \in \mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$.

In fact, we can fix $\alpha \in \Delta_P$ and prove (ii) for all $a \in A_0^+$ with $|\alpha(a)| > \sigma_{R\backslash G}(a)$. As in the proof of Claim 6.2.4, since ξ is nontrivial on $\mathfrak{n}_\alpha(F)$, we can find a constant $C > 0$ such that
$$I(f,a) = 0$$

for all $a \in A_{min}(F)$ with $|\alpha(a)| > C$ and for all $f \in \mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$. Combining with the estimation in part (1), we prove (ii).

By (ii), in order to prove (i), it is enough to prove the following statement

(iii) For all $d > 0$, there exists a continuous seminorm $\nu_{d,K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that

$$|I(f,a)| \ll \nu_{d,K}(f)\Xi^{R\backslash G}(a)^2\sigma_{R\backslash G}(a)^{-d}$$

for all $a \in A_{min}^{a+}$ and $f \in \mathcal{C}_{scusp,K}(Z_G(F)\backslash G(F), \eta^{-1})$.

Claim B.0.2. *In order to prove (iii), it is enough to prove the following statement:*

(iv) For all $d > 0$, there exists a continuous seminorm $\nu_{d,K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that

$$|I(f,a)| \ll \nu_{d,K}(f)\Xi^{R\backslash G}(a)^2\sigma_{R\backslash G}(a)^{-d}$$

for all $a \in A_{min}^{+}$ and $f \in \mathcal{C}_{scusp,K}(Z_G(F)\backslash G(F), \eta^{-1})$.

In fact, by the definition of A_{min}^{a+}, every element $a \in A_{min}^{a+}$ can be written as $a = a_+a_-$ with

$$a_+ \in A_{min}^{+},\ \sigma(a_-) \ll \log(1 + \sigma_{R\backslash G}(a)).$$

Then by (iv), for all $a \in A_{min}^{a+}$, we have

$$|I(f,a)| \leq \nu_{d,K}({}^{a_-}f)\Xi^{R\backslash G}(a_+)\sigma_{R\backslash G}(a_+)^{-d}. \tag{B.1}$$

Then (iii) will follows from the following three inequalities (whose proofs are trivial):

I1 If ν is a continuous seminorm on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$, there exist a continuous seminorm ν' on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ and $c_1 > 0$ such that

$$\nu({}^g f) \leq \nu'(f)e^{c\sigma_0(g)}$$

for all $g \in G(F)$ and $f \in \mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$.

I2 There exists $c_2 > 0$ such that $\Xi^{R\backslash G}(xg) \ll \Xi^{R\backslash G}(x)e^{c_2\sigma_0(g)}$ for all $g \in G(F)$ and $x \in H(F)\backslash G(F)$.

I3 $\sigma_{R\backslash G}(xg)^{-1} \ll \sigma_{R\backslash G}(x)^{-1}\sigma_0(g)$ for all $g \in G(F)$ and $x \in H(F)\backslash G(F)$.

For any maximal parabolic subgroup $\bar{Q} = M_Q U_{\bar{Q}}$ containing $\bar{P}_{min}$ and $\delta > 0$, set

$$A_{min}^{\bar{Q},+}(\delta) = \{a \in A_{min}^+ | \; |\alpha(a)| \geq e^{\delta\sigma_0(a)} \; \forall \alpha \in \Psi(A_{min}, U_{\bar{Q}}).$$

Once we choose δ small, the complement of

$$\cup_{\bar{Q}} A_{min}^{\bar{Q},+}(\delta)$$

in A_{min}^+ is relatively compact modulo the center. Here $\bar{Q}$ runs over all maximal parabolic subgroups containing $\bar{P}_{min}$. Therefore in order to prove (iv), it is enough to prove the following statement:

(v) For all proper maximal parabolic subgroups $\bar{Q}$ containing $\bar{P}_{min}$ and $d > 0$, there exists a continuous seminorm $\nu_{\bar{Q},d,K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that

$$|I(f,a)| \ll \nu_{\bar{Q},d,K}(f)\Xi^{R\backslash G}(a)^2 \sigma_{R\backslash G}(a)^{-d}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $f \in \mathcal{C}_{scusp,K}(Z_G(F)\backslash G(F), \eta^{-1})$.

Now fix a $\bar{Q}$ as in (v), let

$$\mathcal{U}_R = R(F) \cap \bar{P}_{min}(F) U_{min}(F).$$

By the Bruhat decomposition, $\mathcal{U}_R$ is an open subset of $R(F)$ containing the identity element. Let

$$u : \mathcal{U}_R \to U_{min}(F)$$

be the F-analytic map sending $h \in \mathcal{U}_R$ to the unique element $u(h) \in U_{min}(F)$ such that $hu(h)^{-1} \in \bar{P}_{min}(F)$. Since $\bar{P}_{min}$ is a good parabolic subgroup, we have $\bar{\mathfrak{p}}_{min} + \mathfrak{r} = \mathfrak{g}$. Together with the fact that the differential of u at 1 is given by $d_1 u(X) = p_{\mathfrak{u}_{min}}(X)$ where $p_{\mathfrak{u}_{min}}$ is the linear projection of $\mathfrak{g}$ onto $\mathfrak{u}_{min}$ relative to the decomposition $\mathfrak{g} = \bar{\mathfrak{p}}_{min} \oplus \mathfrak{u}_{min}$, we know that the map u is submersive at the identity element. Therefore we can find a relatively compact open neighborhood $\mathcal{U}_{min}$ of 1 in $U_{min}(F)$ and an F-analytic section

$$h : \mathcal{U}_{min} \to \mathcal{U}_R$$

$$u \to h(u)$$

of the map u such that $h(1)=1$. Without loss of generality, we assume that the levi component M_Q of $\bar{Q}$ contains M_{min}, and let $Q=M_QU_Q$ be the opposite parabolic subgroup of $\bar{Q}$ with respect to M_Q. Set

$$\mathcal{U}_Q=\mathcal{U}_{min}\cap U_Q(F),\ R_{\bar{Q}}=R\cap \bar{Q},\ \mathcal{U}_{R,Q}=R_{\bar{Q}}(F)h(\mathcal{U}_Q).$$

It is easy to see that the map

$$R_{\bar{Q}}(F)\times \mathcal{U}_Q\rightarrow R(F):\ (h_{\bar{Q}},u_Q)\rightarrow h_{\bar{Q}}h(u_Q)$$

is an injective F-analytic local isomorphism. Hence its image $\mathcal{U}_{R,Q}$ is an open subset of $H(F)$ containing the identity element. Let j be the Jacobian of this map, it is a smooth function on $R_{\bar{Q}}(F)\times\mathcal{U}_Q$ and it is obviously invariant under the $R_{\bar{Q}}(F)$-translation. For simplicity, we write $j(u_Q)=j(h_{\bar{Q}},u_Q)$. Therefore for all $\varphi\in L^1(\mathcal{U}_{R,Q})$, we have

$$\int_{\mathcal{U}_{R,Q}}\varphi(h)dh=\int_{R_{\bar{Q}}(F)}\int_{\mathcal{U}_Q}\varphi(h_{\bar{Q}}h(u_Q))j(u_Q)du_Qdh_{\bar{Q}}. \tag{B.2}$$

Fix $\epsilon>0$ small, we need the following statement:

(vi) Let $0<\delta'<\delta$ and $c_0>0$. If ϵ is small enough, we have

$$aU_Q[<\epsilon\sigma_0(a)]a^{-1}\subset \exp(B(0,c_0e^{-\delta'\sigma_0(a)})\cap\mathfrak{u}_Q(F))$$

for all $a\in A_{min}^{\bar{Q},+}(\delta)$.

In fact, if $\sigma_0(a)\leq\epsilon^{-1}$, the left hand side is empty, hence (vi) holds. If $\sigma_0(a)>\epsilon^{-1}$, we can find $\alpha>0$ such that

$$|\log(u)|\leq e^{\alpha\sigma(u)}$$

for all $u\in U_Q(F)$. We can also find $\beta>0$ such that

$$|aXa^{-1}|\leq \beta e^{-\delta\sigma_0(a)}|X|$$

for all $X\in\mathfrak{u}_Q(F)$ and $a\in A_{min}^{\bar{Q},+}(\delta)$. As a result, for $\epsilon>0$, we have

$$\begin{aligned}&|\log(aua^{-1})|=|a\log(u)a^{-1}|\leq\beta e^{-\delta\sigma_0(a)}|\log(u)|\\ \leq\ &\beta e^{(a\epsilon-\delta)\sigma_0(a)}=\beta e^{(a\epsilon+\delta'-\delta)\sigma_0(a)}e^{-\delta'\sigma_0(a)}\end{aligned}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $u \in U_Q[<\epsilon\sigma_0(a)]$. Then we only need to choose ϵ small enough such that $\beta e^{(a\epsilon+\delta'-\delta)\sigma_0(a)} \leq c_0$ for all $a \in A_{min}(F)$ with $\sigma_0(a) > \epsilon^{-1}$. This proves (vi).

By (vi), for ϵ small, we have

$$aU_Q[<\epsilon\sigma_0(a)]a^{-1} \subset \mathcal{U}_Q$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$. Fix such ϵ, we define

$$\mathcal{U}_{R,\bar{Q}}^{\epsilon,a} = R_{\bar{Q}}[\sigma_0 < \epsilon\sigma_0(a)]h(aU_Q[<\epsilon\sigma_0(a)]a^{-1}).$$

Then (v) will be a consequence of the following two statements:

(vii) For all $d > 0$, there exists a continuous seminorm $\nu_{d,K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that

$$\int_{Z_R(F)\backslash(R(F)\backslash\mathcal{U}_{R,\bar{Q}}^{\epsilon,a})} |f(a^{-1}ha)|dh \leq \nu_{d,K}(f)\Xi^{R\backslash G}(a)^2\sigma_{R\backslash G}(a)^{-d}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $f \in \mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$.

(viii) For all $d > 0$, there exists a continuous seminorm $\nu_{d,K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that

$$\left|\int_{Z_R(F)\backslash\mathcal{U}_{R,\bar{Q}}^{\epsilon,a}} f(a^{-1}ha)\omega(h)\xi(h)dh\right| \leq \nu_{d,K}(f)\Xi^{R\backslash G}(a)^2\sigma_{R\backslash G}(a)^{-d}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $f \in \mathcal{C}_{scusp,K}(Z_G(F)\backslash G(F), \eta^{-1})$.

We first prove (vii), we need a claim.

Claim B.0.3. *For all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $h \in R(F)\backslash\mathcal{U}_{R,\bar{Q}}^{\epsilon,a}$, we have*

$$\sigma_0(a) \ll \sigma_0(a^{-1}ha).$$

In fact, by Lemma 1.3.1 of [B15] and Proposition 4.2.1(3), it is enough to show that we can find $\epsilon' > 0$ such that

$$R(F) \cap (\bar{Q}[\sigma_0 < \epsilon'\sigma_0(a)]aU_Q[<\epsilon'\sigma_0(a)]a^{-1}) \subset \mathcal{U}_{R,\bar{Q}}^{\epsilon,a} \tag{B.3}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$. Fix $\epsilon' > 0$ small, let $a \in A_{min}^{\bar{Q},+}(\delta)$. If $\sigma_0(a) \leq (\epsilon')^{-1}$, the left hand side of (B.3) is empty and there is nothing to prove. If $\sigma_0(a) > (\epsilon')^{-1}$, we assume that $\epsilon' < \epsilon$. Let $h \in R(F) \cap (\bar{Q}[\sigma_0 < \epsilon'\sigma_0(a)]aU_Q[< \epsilon'\sigma_0(a)]a^{-1})$. We have

$$aU_Q[< \epsilon'\sigma_0(a)]a^{-1} \subset aU_Q[< \epsilon\sigma_0(a)]a^{-1} \subset \mathcal{U}_Q.$$

Let $h = qu$ with $q \in \bar{Q}[\sigma_0 < \epsilon'\sigma_0(a)]$ and $u \in aU_Q[< \epsilon'\sigma_0(a)]a^{-1} \subset \mathcal{U}_Q$. By the definition of the map h, $uh(u)^{-1} = (h(u)u^{-1})^{-1} \in \bar{P}_{min}(F) \subset \bar{Q}(F)$. Hence $h = q(uh(u)^{-1})h(u)$ with $q(uh(u)^{-1}) \in R(F) \cap \bar{Q}(F) = R_{\bar{Q}}(F)$. Therefore we can find $u \in aU_Q[< \epsilon'\sigma_0(a)]a^{-1}$ such that $hh(u)^{-1} \in R_{\bar{Q}}(F)$. By the definition of $\mathcal{U}_{R,\bar{Q}}^{\epsilon,a}$, in order to prove (B.3), we only need to show that if ϵ' is small enough, we have

$$\sigma_0(hh(u)^{-1}) < \epsilon\sigma_0(a). \tag{B.4}$$

By (vi), if ϵ' is small enough, the sets $aU_Q[< \epsilon'\sigma_0(a)]a^{-1}$ remain in a fixed compact subset as a varies in $A_{min}^{\bar{Q},+}(\delta)$. Hence $h(u)$ is uniformly bounded which is independent of a and h. This implies $\sigma(h(u)) \ll 1 \ll \epsilon'\sigma_0(a)$ since $\sigma_0(a) > (\epsilon')^{-1}$. Therefore

$$\sigma_0(hh(u)^{-1}) \ll \sigma_0(h) + \sigma(h(u)) \ll \epsilon'\sigma_0(a).$$

This proves (B.4), and finishes the proof of Claim B.0.3.

By the claim above, given $d > 0$, for all $d' > 0$, we have

$$\int_{Z_R(F)\backslash(R(F)\backslash\mathcal{U}_{R,\bar{Q}}^{\epsilon,a})} |f(a^{-1}ha)|dh \ll q_{d'}(f)\sigma_0(a)^{-d'/2} \int_{Z_R(F)\backslash R(F)} \Xi^G(a^{-1}ha)\sigma_0(a^{-1}ha)^{-d'/2}dh$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $f \in \mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$. By Proposition 4.4.1(5) and Lemma 4.2.6(2), for d' large, the right hand side above is essentially bounded by

$$q_{d'}(f)\sigma_0(a)^{-d}\Xi^{R\backslash G}(a)^2$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$. This proves (vii).

Now the only thing left is to prove (viii). By (B.2), we have

$$\int_{Z_R(F)\backslash\mathcal{U}_{R,\bar{Q}}^{\epsilon,a}} f(a^{-1}ha)\omega(h)\xi(h)dh = \int_{Z_R(F)\backslash R_{\bar{Q}}[\sigma_0<\epsilon\sigma_0(a)]} \tag{B.5}$$

$$\cdot \int_{aU_Q[<\epsilon\sigma_0(a)]a^{-1}} f(a^{-1}h_{\bar{Q}}h(u_Q)a)\omega(h_{\bar{Q}}h(u_Q))\xi(h_{\bar{Q}}h(u_Q))j(u_Q)du_Q dh_{\bar{Q}}$$

for all $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ and $a \in A_{min}^{\bar{Q},+}(\delta)$. Without loss of generality, we assume that $j(1) = 1$. Every $h_{\bar{Q}} \in R_{\bar{Q}}(F)$ can be written as $h_{\bar{Q}} = u_{\bar{Q}}(h_{\bar{Q}})m_Q(h_{\bar{Q}})$ with $u_{\bar{Q}}(h_{\bar{Q}}) \in U_{\bar{Q}}$ and $m_Q(h_{\bar{Q}}) \in M_Q$. We need a lemma.

Lemma B.0.4. *Let $0 < \delta' < \delta$ and $d' > 0$. There exists a continuous semi-norm $\mu_{d',K}$ on $\mathcal{C}_K(Z_G(F)\backslash G(F), \eta^{-1})$ such that if ϵ is small enough, we have*

$$|\omega \otimes \xi(h(u_Q))j(u_Q) - 1| = 0 \tag{B.6}$$

and

$$|f(a^{-1}ha) - f(a^{-1}m_Q(h_{\bar{Q}})u_Q a)| = 0 \tag{B.7}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$, $u_Q \in aU_Q[< \epsilon\sigma_0(a)]a^{-1}$ and $h_{\bar{Q}} \in R_{\bar{Q}}[\sigma_0 < \epsilon\sigma_0(a)]$. Here $h = h_{\bar{Q}}h(u_Q)$.

Proof. We first prove (B.6). Since the functions $(\omega \otimes \xi) \circ h \circ \exp$ and $j \circ \exp$ are smooth functions on $\log(\mathcal{U}_Q) \subset \mathfrak{u}_Q(F)$, we can choose a compact neighborhood $\omega_Q \subset \log(\mathcal{U}_Q)$ of 0 such that the two functions above are constant on ω_Q. By (vi), if ϵ is small enough, for all $a \in A_{min}^{\bar{Q},+}(\delta)$, we have

$$aU_Q[< \epsilon\sigma_0(a)]a^{-1} \subset \exp(\omega_Q). \tag{B.8}$$

Therefore the left hand side of (B.6) is always 0, and this proves (B.6).

Now we prove (B.7). Let $\omega_G \subset \mathfrak{g}(F)$ be a compact neighborhood of 0 on which the exponential map is well defined and we have $\exp(\omega_G) \subset K$. For $h_{\bar{Q}} \subset R_{\bar{Q}}(F)$, $u_Q \in \mathcal{U}_Q$ and $a \in A_{min}(F)$, we have

$$a^{-1}m_Q(h_{\bar{Q}})u_Q a = k_1^{-1}a^{-1}hak_2^{-1}$$

where $h = h_{\bar{Q}}h(u_Q)$, $k_1 = a^{-1}u_{\bar{Q}}(h_{\bar{Q}})a$ and $k_2 = a^{-1}u_Q^{-1}h(u_Q)a$. Since f is bi-K-invariant, in order to prove (B.7), it is enough to prove the following claim.

Claim B.0.5. *Let $0 < \delta' < \delta$. Then if ϵ small enough, we have*

$$a^{-1}u_{\bar{Q}}(h_{\bar{Q}})a \in \exp(B(0, e^{-\delta'\sigma_0(a)}) \cap \omega_G) \tag{B.9}$$

and

$$a^{-1}u_Q^{-1}h(u_Q)a \in \exp(B(0, e^{-\delta'\sigma_0(a)}) \cap \omega_G) \tag{B.10}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$, $u_Q \in aU_Q[< \epsilon\sigma_0(a)]a^{-1}$ and $h_{\bar{Q}} \in R_{\bar{Q}}[\sigma_0 < \epsilon\sigma_0(a)]$.

The proof of (B.9) is the same as the proof of (vi), we will skip it here. For (B.10), let $\bar{p}_{min}(u) = h(u)u^{-1}$ for all $u \in \mathcal{U}_{min}$. It defines an F-analytic map from $\mathcal{U}_{min}$ to $\bar{P}_{min}(F)$, and we have

$$a^{-1}u_Q^{-1}h(u_Q)a = a^{-1}u_Q^{-1}\bar{p}_{min}(u_Q)u_Q a \tag{B.11}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $u_Q \in aU_Q[<\epsilon\sigma_0(a)]a^{-1}$. Since $\bar{p}_{min}(1) = 1$, there exists an open neighborhood $\mathcal{U}'_Q \subset \mathcal{U}_Q$ of 1 and an F-analytic map $u_Q \in \mathcal{U}'_Q \mapsto X(u_Q) \in \bar{\mathfrak{p}}_{min}(F)$ such that

$$\bar{p}_{min}(u_Q) = e^{X(u_Q)}$$

for all $u_Q \in \mathcal{U}'_Q$. Applying (vi) again, we know that for ϵ small enough, we have $aU_Q[<\epsilon\sigma_0(a)]a^{-1} \subset \mathcal{U}'_Q$ for all $a \in A_{min}^{\bar{Q},+}(\delta)$. Therefore (B.11) becomes

$$a^{-1}u_Q^{-1}h(u_Q)a = e^{Ad(a^{-1}u_Q^{-1})X(u_Q)}.$$

Hence in order to prove (B.10), we only need to show that if ϵ is small enough, we have

$$Ad(a^{-1}u_Q^{-1})X(u_Q) \in B(0, e^{-\delta'\sigma_0(a)}) \cap \omega_G \tag{B.12}$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $u_Q \in aU_Q[<\epsilon\sigma_0(a)]a^{-1}$.

There exists $\alpha > 0$ such that

$$|Ad(g^{-1})X| \leq e^{\alpha\sigma_0(g)}|X|$$

for all $g \in G(F)$ and $X \in \mathfrak{g}(F)$. Hence we have

$$|Ad(a^{-1}u_Q^{-1})X(u_Q)| = |Ad(a^{-1}u_Q^{-1}a)Ad(a^{-1})X(u_Q)| \leq e^{\alpha\epsilon\sigma_0(a)}|Ad(a^{-1})X(u_Q)|$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $u_Q \in aU_Q[<\epsilon\sigma_0(a)]a^{-1}$. Moreover, by the definition of A_{min}^{+}, there exists $\beta > 0$ such that

$$|Ad(a^{-1})X| \leq \beta|X|$$

for all $a \in A_{min}^{+}$ and $X \in \bar{\mathfrak{p}}_{min}(F)$. Therefore we have

$$|Ad(a^{-1}u_Q^{-1})X(u_Q)| \leq e^{\alpha\epsilon\sigma_0(a)}|Ad(a^{-1})X(u_Q)| \leq \beta e^{\alpha\epsilon\sigma_0(a)}|X(u_Q)|$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$ and $u_Q \in aU_Q[<\epsilon\sigma_0(a)]a^{-1}$. So in order to prove (B.12), we only need to show that if ϵ is small enough, we have

$$X(aU_Q[<\epsilon\sigma_0(a)]a^{-1}) \subset \beta^{-1}e^{-\alpha\epsilon\sigma_0(a)}(B(0, e^{-\delta'\sigma_0(a)}) \cap \omega_G)$$

for all $a \in A_{min}^{\bar{Q},+}(\delta)$. This just follows from (vi) and the fact that the map $X(\cdot)$ is an analytic map. This finishes the proof of the lemma. $\square$

Combining the lemma above and (B.5), we conclude that in order to prove (viii), it is enough to prove the following statement:

(ix) For all $d > 0$, there exists a continuous seminorm ν_d on $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ such that

$$\begin{aligned} & |\int_{Z_R(F)\backslash R_{\bar{Q}}[\sigma_0<\epsilon\sigma_0(a)]} \int_{aU_Q[<\epsilon\sigma_0(a)]a^{-1}} f(a^{-1}m_Q(h_{\bar{Q}})u_Q a)\omega(h_{\bar{Q}})\xi(h_{\bar{Q}})du_Q dh_{\bar{Q}}| \\ \leq \ & \nu_d(f)\Xi^{R\backslash G}(a)^2\sigma_{R\backslash G}(a)^{-d} \end{aligned}$$

for all $a \in A_{min}^+$ and $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$.

Denote by $I_{\bar{Q}}^{\epsilon}(f,a)$ the integral above. By changing the variable $u_Q \to au_Q a^{-1}$, we have

$$I_{\bar{Q}}^{\epsilon}(f,a) = \delta_Q(a)\int_{Z_R(F)\backslash R_{\bar{Q}}[\sigma_0<\epsilon\sigma_0(a)]} \int_{U_Q[<\epsilon\sigma_0(a)]} f(a^{-1}m_Q(h_{\bar{Q}})au_Q)du_Q\omega(h_{\bar{Q}})\xi(h_{\bar{Q}})dh_{\bar{Q}}.$$

Since f is strongly cuspidal, we have

$$\int_{U_Q[<\epsilon\sigma_0(a)]} f(a^{-1}m_Q(h_{\bar{Q}})au_Q)du_Q = -\int_{U_Q[\geq\epsilon\sigma_0(a)]} f(a^{-1}m_Q(h_{\bar{Q}})au_Q)du_Q.$$

For $d_1 > 0$, the integral above is bounded by

$$q_{d_1}(f)\int_{U_Q[\geq\epsilon\sigma_0(a)]} \Xi^G(a^{-1}m_Q(h_{\bar{Q}})au_Q)\sigma_0(a^{-1}m_Q(h_{\bar{Q}})au_Q)^{-d_1}du_Q. \tag{B.13}$$

Since $\sigma_0(m_Q u_Q) \gg \sigma_0(u_Q)$ for all $m_Q \in M_Q(F)$ and $u_Q \in U_Q(F)$, for all $d_2 > 0$, (B.13) is essentially bounded by

$$q_{d_1}(f)\sigma_0(a)^{-d_2}\int_{U_Q[\geq\epsilon\sigma_0(a)]} \Xi^G(a^{-1}m_Q(h_{\bar{Q}})au_Q)\sigma_0(a^{-1}m_Q(h_{\bar{Q}})au_Q)^{-d_1+d_2}du_Q.$$

For $d_3 > 0$, by Proposition 2.8.3, if d_1 is large enough, the integral above is essentially bounded by

$$\delta_{\bar{Q}}(m_Q(h_{\bar{Q}}))\Xi^{M_Q}(a^{-1}m_Q(h_{\bar{Q}})a)\sigma_0(a^{-1}m_Q(h_{\bar{Q}})a)^{-d_3}.$$

Therefore for such d_1, $|I_{\bar{Q}}^{\epsilon}(f,a)|$ is essentially bounded by

$$\delta_Q(a)q_{d_1}(f)\sigma_0(a)^{-d_2}\int_{Z_R(F)\backslash R_{\bar{Q}}(F)} \delta_{\bar{Q}}(m_Q(h_{\bar{Q}}))\Xi^{M_Q}(a^{-1}m_Q(h_{\bar{Q}})a)\sigma_0(a^{-1}m_Q(h_{\bar{Q}})a)^{-d_3}dh_{\bar{Q}} \tag{B.14}$$

for all $a \in A_{min}$ and $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$. Let $G_Q = \bar{Q}/U_{\bar{Q}}$, it can be identified with M_Q. Since $R \cap U_{\bar{Q}} = \{1\}$ by Proposition 4.2.1, $R_{\bar{Q}}$ can be identified with a subgroup of $G_{\bar{Q}}$ as in Chapter 4. Then (B.14) becomes

$$\delta_Q(a) q_{d_1}(f) \sigma_0(a)^{-d_2} \int_{Z_R(F)\backslash R_{\bar{Q}}} \delta_{\bar{Q}}(h_{\bar{Q}}) \Xi^{G_Q}(a^{-1} h_{\bar{Q}} a) \sigma_0(a^{-1} h_{\bar{Q}} a)^{-d_3} dh_{\bar{Q}}.$$

By Lemma 4.4.2(1) and (3), if d_3 is large enough, the last term above is essentially bounded by

$$\delta_Q(a) q_{d_1}(f) \sigma_0(a)^{-d_2} \Xi^{G_Q}(a)^2$$

for all $a \in A_{min}^+$. By Proposition 2.8.3(1), Lemma 4.2.6(2) and Proposition 4.4.1(2), there exists $d_4 > 0$ such that

$$\delta_Q(a) \Xi^{G_Q}(a)^2 \ll \Xi^{R\backslash G}(a)^2 \sigma_0(a)^{d_4}$$

for all $a \in A_{min}^+$. Once we take $d_2 = d + d_4$, we know that for d_1 large enough, we have

$$|I_{\bar{Q}}^{\epsilon}(f, a)| \ll q_{d_1}(f) \Xi^{R\backslash G}(a)^2 \sigma_0(a)^{-d}$$

for all $a \in A_{min}^+$ and $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$. Then (ix) will follows from Lemma 4.2.6(2).

Now the proof of Proposition 8.1.1 is finally complete.

Appendix C

The Reduced Models

In this appendix, we will summarize our results for the reduced models of the Ginzburg-Rallis model. The proof of these results are similar to the Ginzburg-Rallis model case we considered in this paper, we will skip the details. For simplicity, we will use (G,R) instead of $(G_{\bar{Q}}, R_{\bar{Q}})$ to represent the reduced models. For any irreducible admissible generic representation π of $G(F)$, we use $m(\pi)$ to denote the multiplicity for the reduced model.

C.1 Type II Models

As mentioned in Section 5.4, if (G,R) is a Type II reduced model, the geometric side of the trace only contains the germ at the identity element. Therefore the multiplicity formula for the model (G,R) is just

$$m(\pi) = m_{geom}(\pi) := c_{\theta_\pi, \mathcal{O}_{reg}}(1).$$

In particular, by the work of Rodier, we know that the multiplicity $m(\pi)$ is always 1.

C.2 Trilinear GL_2 Model

In this section, let (G,H) and (G_D, H_D) be the trilinear GL_2 models introduced in Section 4.5. We use $m(\pi)$ and $m(\pi_D)$ to denote the multiplicities. Then by applying our methods in this paper, we can prove the following theorem.

Theorem C.2.1. *If π is an irreducible tempered representation of $G(F)$ whose central character equals χ^2 on $Z_H(F)$, let π_D be the Jacquet-Langlands correspondence of π to G_D if it exist; otherwise let $\pi_D = 0$. Then we have*

$$m(\pi) + m(\pi_D) = 1.$$

Remark C.2.2. *If F is p-adic or $\mathbb{R}$, the above Theorem has been proved by Prasad [P90] and Loke [L01] for general generic representations by using different methods. In the loc. cit., they also proved the epsilon dichotomy conjecture for this model. In [L01], the author also proved the complex case for generic representations satisfy certain assumption.*

Moreover, if F is p-adic, we can also prove the relative trace formulas for this model and the multiplicity formulas for $m(\pi)$ and $m(\pi_D)$. In particular, we can show that the multiplicity formula

$$m(\pi) = m_{geom}(\pi) := \sum_{T\in\mathcal{T}} \mid W(H,T) \mid^{-1} \nu(T) \int_{Z_G(F)\backslash T(F)} c_\pi(t) D^H(t) \chi(\det(t))^{-1} dt$$

holds for all tempered representations π of $G(F)$. Here $c_\pi(t)$ is the germ associated to θ_π defined in Section 5.4. Similarly, we can also prove the multiplicity formula for $m(\pi_D)$.

Remark C.2.3. *In fact, we can show that the multiplicity formulas above hold for all generic representations. We first consider the split case. If $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ is an essentially discrete series of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, by twisting π by some characters, we may assume that π is a discrete series. Note that this is allowable since both $m(\pi)$ and $m_{geom}(\pi)$ are invariant under the unramified twist. This proves the multiplicity formula when π is an essentially discrete series. If π is not an essentially discrete series, then one of the π_i is a principal series. By the work of Prasad in [P90], we know that the multiplicity equals 1 in this case. On the other hand, by Lemma 3.3.1, the germ $c_\pi(t)$ equals zero for all $t \in T_v(F)_{reg}$ and $v \in F^\times/(F^\times)^2$ with $v \neq 1$. Therefore $m_{geom}(\pi) = c_\pi(1) = 1$. This proves the multiplicity formula.*

If we are in the quaternion case, every irreducible representation π_D of $G_D(F)$ is an essential discrete series. So we only need to twist π_D by some characters and then apply our results for the discrete series.

C.3 The Generalized Trilinear GL_2 Models

In this section, we consider the generalized trilinear GL_2 models. Although these models are not the reduced models for the Ginzburg-Rallis model, they are very similar to the trilinear GL_2 model case we considered in the previous section, hence our methods in this paper can also be applied to these models. These models were first considered by Prasad in [P92] for general generic representations using different methods. By using our method in this paper, we can prove the tempered case. In this section, F is a p-adic field.

Case I: Let K/F be a cubic field extension, $G(F) = \mathrm{GL}_2(K)$, and $H(F) = \mathrm{GL}_2(F)$. On the mean time, let $G_D(F) = \mathrm{GL}_1(D_K)$ and $H_D(F) = \mathrm{GL}_1(D)$ where $D_K = D \otimes_F K$. For a given irreducible representation π of $G(F)$, assume that the restriction of the central character $\omega_\pi : K^\times \to \mathbb{C}^\times$ to $F^\times$ equals χ^2 for some character χ of $F^\times$. χ will induces a one-dimensional representation σ of $H(F)$. Let

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi, \sigma). \tag{C.1}$$

Similarly we can define $m(\pi_D)$ for an irreducible representation π_D of $G_D(F)$. The following theorem has been proved by Prasad in [P92] for general generic representation using different method. By using our method in this paper, we can prove the tempered case.

Theorem C.3.1. *If π is a tempered representation of G, let π_D be the Jacquet-Langlands correspondence of π to G_D if it exist; otherwise let $\pi_D = 0$. Then*

$$m(\pi) + m(\pi_D) = 1.$$

We can also prove the relative trace formulas for this model and the multiplicity formulas for $m(\pi)$ and $m(\pi_D)$. In particular, we can show that the multiplicity formula

$$m(\pi) = m_{geom}(\pi) := \sum_{T \in \mathcal{T}} \mid W(H,T) \mid^{-1} \nu(T) \int_{Z_G(F) \backslash T(F)} c_\pi(t) D^H(t) \chi(\det(t))^{-1} dt$$

holds for all tempered representations π of $G(F)$. Here $c_\pi(t)$ is the germ associated to θ_π defined in the same way as the trilinear GL_2 model case. Similarly, we can also prove the multiplicity formula for $m(\pi_D)$. Moreover, by the same argument as in Remark C.2.3

together with Prasad's results in [P92], we can show that the multiplicity formulas above hold for all generic representations.

Case II: Let $E = F_v$ be a quadratic extension of F where v is a non-trivial square class in $F^{\times}$. Let $G(F) = \mathrm{GL}_2(E) \oplus \mathrm{GL}_2(F)$, $H(F) = \mathrm{GL}_2(F)$, $G_D(F) = \mathrm{GL}_2(E) \times \mathrm{GL}_1(D)$ and $H_D(F) = \mathrm{GL}_1(D)$. As in the previous cases, we can define the multiplicity $m(\pi)$ (resp. $m(\pi_D)$) for the model $(G(F), H(F))$ (resp. $(G_D(F), H_D(F))$). By using our method in this paper, we can still prove that the summation of the multiplicities over any tempered L-packet is 1. We can also prove the relative trace formulas and the multiplicity formulas. Moreover, by the same argument as in Remark C.2.3 together with Prasad's results in [P92], we can also show that the multiplicity formulas hold for all generic representations. However, there is one difference between this case and all the previous cases, this will be discussed in the following remark.

Remark C.3.2. *In all the previous cases, for the geometric side of the trace formulas (or the multiplicity formulas), we are integrating the germs of the distribution over all nonsplit tori of $H(F)$. But in this case, we only need to integrate over those nonsplit tori which is not isomorphic to T_v. The reason is that in this case, both $G(F)$ and $G_D(F)$ contain $\mathrm{GL}_2(E)$. As a result, for an element in $T_v(F) \cap H(F)_{reg}$ (or $T_v(F) \cap H_D(F)_{reg}$), although it is elliptic in $H(F)$ and $H_D(F)$, it will no longer be elliptic in $G(F)$ or $G_D(F)$. Therefore the localization at this element will be zero. This is why the torus T_v will not show up in the multiplicity formulas and the geometric side of the relative trace formulas.*

C.4 The Middle Models

In this section, let (G, R) and (G_D, R_D) be the middle models introduced in Section 4.5. We use $m(\pi)$ and $m(\pi_D)$ to denote the multiplicities. Then by applying our methods in this paper, we can prove the following theorem.

Theorem C.4.1. *If π is an irreducible tempered representation of $G(F)$ whose central character equals χ^2 on $Z_H(F)$, let π_D be the Jacquet-Langlands correspondence of π to G_D if it exist; otherwise let $\pi_D = 0$. Then we have*

$$m(\pi) + m(\pi_D) = 1.$$

Conjecture C.4.2. *In general, we expect that the above theorem holds for all generic representations.*

We can also prove the epsilon dichotomy conjecture for this case. We need some preparation, let $\pi = \pi_1 \otimes \pi_2$ be an irreducible generic representation of $G(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$. Let ω_{π_1} (resp. ω_{π_1}) be the central character of π_1 (resp. π_2). As in the Ginzburg-Rallis model case, we assume that $\omega_{\pi_1}\omega_{\pi_2} = 1$. Let ϕ_1 (resp. ϕ_2) be the Langlands parameter of π_1 (resp. π_2). Then we have

$$\wedge^3(\phi_1 \oplus \phi_2) = (\wedge^2(\phi_1) \otimes \phi_2) \oplus (\wedge^3(\phi_1)) \oplus (\phi_1 \otimes (\det(\phi_2))).$$

Since $\det(\phi_1)\det(\phi_2) = 1$, we have

$$(\wedge^3(\phi_1))^\vee = \det(\phi_1)^{-1} \otimes \phi_1 = \phi_1 \otimes (\det(\phi_2)).$$

This implies

$$\epsilon(1/2, \wedge^3(\phi_1))\epsilon(1/2, \phi_1 \otimes (\det(\phi_2))) = \det(\wedge^3(\phi_1))(-1) = \omega_{\pi_1}(-1).$$

Hence the multiplicity is related to the epsilon factor

$$\omega_{\pi_1}(-1)\epsilon(1/2, \pi_1, \wedge^2)\epsilon(1/2, \pi_2).$$

The following conjecture is the epsilon dichotomy conjecture for the middle model.

Conjecture C.4.3. *With the notations and the assumptions above, the followings hold.*

$$m(\pi) = 1 \iff \omega_{\pi_1}(-1)\epsilon(1/2, \pi_1, \wedge^2)\epsilon(1/2, \pi_2) = 1,$$

$$m(\pi) = 0 \iff \omega_{\pi_1}(-1)\epsilon(1/2, \pi_1, \wedge^2)\epsilon(1/2, \pi_2) = -1.$$

Our results for the conjecture above can be summarized in the following theorem.

Theorem C.4.4. *Assume that π is tempered. The followings hold.*

1. *If F is archimedean, then Conjecture C.4.3 holds.*

2. *If F is p-adic and if π is not a discrete series, then Conjecture C.4.3 holds.*

Proof. The proof is similar to the Ginzburg-Rallis model case in Chapter 7 and 13. In other word, if $F = \mathbb{C}$, by the same argument as in Section 7.1, we can show that the epsilon factor is always equal to 1. Then by applying Theorem C.4.1 above, we know that the multiplicity is also equal to 1, this proves the conjecture.

If $F = \mathbb{R}$, then by the same argument as in Section 7.3, we can reduce the problem to the trilinear GL_2 model case. Then the conjecture will follows from the work of Prasad [P90] and Loke [L01].

Finally if F is p-adic, by our assumption, there are two possibilities: either π is induced from the trilinear GL_2 model or π is induced from some Type II model. If π is induced from the trilinear GL_2 model, we can again reduce the problem to the trilinear GL_2 model case and then applying Prasad's result in [P90]. If π is induced from some Type II model, then $\pi_D = 0$. By Theorem C.4.1 above, we know that $m(\pi) = 1$. By the same argument as in Section 13.3, we can show that the epsilon factor is also equal to 1 in this case, and this proves the conjecture. □

Remark C.4.5. *Assume that F is p-adic. If the central characters of π_1 and π_2 are both trivial, we can find a representation Π of $\mathrm{SO}(6) \times \mathrm{SO}(3)$ associated to π. Then it is easy to see that the multiplicity $m(\pi)$ is equal to the multiplicity $m(\Pi)$ for the Gan-Gross-Prasad model. Also one can show that the epsilon factor associated to π is equal to $\epsilon(1/2, \Pi)$. Then by applying the work of Moeglin and Waldspurger for the Gan-Gross-Prasad model in [MW12], we know that Conjecture C.4.2 and Conjecture C.4.3 hold for all generic representations π with $\omega_{\pi_1} = \omega_{\pi_2} = 1$.*

Moreover, if F is p-adic, we can also prove the relative trace formulas for this model and the multiplicity formulas for $m(\pi)$ and $m(\pi_D)$. In particular, we can show that the multiplicity formula

$$m(\pi) = m_{geom}(\pi) := \sum_{T \in \mathcal{T}} \mid W(H,T) \mid^{-1} \nu(T) \int_{Z_G(F)\backslash T(F)} c_\pi(t) D^H(t) \Delta_Q(t) \chi(\det(t))^{-1} dt$$

holds for all tempered representations π of $G(F)$. Here $c_\pi(t)$ is the germ associated to θ_π defined in Section 5.4, and $\Delta_Q(t)$ is some normalized function also defined in Section 5.4. Similarly, we can also prove the multiplicity formula for $m(\pi_D)$.

Finally, as in Chapter 14, if F is archimedean, we will have some partial results for the general generic representations. We first consider the case when $F = \mathbb{R}$. Let

$\pi = \pi_1 \otimes \pi_2$ be a generic representation of $G(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, and π_D be its Jacquet-Langlands correspondence to $G_D(F)$. By the Langlands classification, there is a parabolic subgroup $Q = LU_Q$ of $\mathrm{GL}_4(F)$ containing the lower Borel subgroup and an essential tempered representation $\tau = \otimes_{i=1}^{k} \tau_i| \;|^{s_i}$ of $L(F)$ with τ_i tempered, $s_i \in \mathbb{R}$ and $s_1 < s_2 < \cdots < s_k$ such that $\pi_1 = I_Q^{\mathrm{GL}_4(F)}(\tau)$. We say Q is nice if $Q \subset \bar{P}_{2,2}$ or $\bar{P}_{2,2} \subset Q$. Here $\bar{P}_{2,2}$ is the parabolic subgroup of $\mathrm{GL}_4(F)$ of type $(2,2)$ and containing the lower Borel subgroup. Then our results can be summarized in the following theorem.

Theorem C.4.6. *With the notations above, the followings hold.*

1. *If $\pi_D = 0$, assume that Q is nice, then Conjecture C.4.2 and Conjecture C.4.3 hold.*

2. *If $\pi_D \neq 0$, we have*
$$m(\pi) + m(\pi_D) \geq 1.$$
Moreover if $\omega_{\pi_1}\omega_{\pi_2} = 1$ (as in Conjecture C.4.3), we have
$$\omega_{\pi_1}(-1)\epsilon(1/2, \pi_1, \wedge^2)\epsilon(1/2, \pi_2) = 1 \Rightarrow m(\pi) = 1,$$
$$m(\pi) = 0 \Rightarrow \omega_{\pi_1}(-1)\epsilon(1/2, \pi_1, \wedge^2)\epsilon(1/2, \pi_2) = -1.$$

As in the Ginzburg-Rallis model case, the assumption on Q can be removed if we can prove the holomorphic continuation of certain generalized Jacquet integrals (i.e. the hypothesis in Section 14.3).

Then we consider the case when $F = \mathbb{C}$. Still let $\pi = \pi_1 \otimes \pi_2$ be a generic representation of $G(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$. As in the Ginzburg-Rallis model case, we know that Conjecture C.4.3 will follows from Conjecture C.4.2. For Conjecture C.4.2, let $B = M_0U_0 \subset \mathrm{GL}_4(F)$ be the Borel subgroup consists of all the lower triangular matrix, here $M_0 = (\mathrm{GL}_1)^4$ is just the group of diagonal matrices. Then π_1 is of the form $I_B^G(\chi)$ where $\chi = \otimes_{i=1}^{4} \chi_i$ is a character on M_0. For $1 \leq i \leq 4$, we can find an unitary character σ_i and some real number $s_i \in \mathbb{R}$ such that $\chi_i = \sigma_i| \;|^{s_i}$. Without loss of generality, we assume that $s_i \leq s_j$ for any $i \geq j$. Then if we combine those representations with the same exponents s_i, we can find a parabolic subgroup $Q = LU_Q$ containing B with $L = \times_{i=1}^{k} \mathrm{GL}_{n_i}$, a representation $\tau = \otimes_{i=1}^{k} \tau_i| \;|^{t_i}$ of $L(F)$ where τ_i are all tempered and the exponents t_i are strictly increasing (i.e. $t_1 < t_2 < \cdots < t_k$)

such that $\pi = I_Q^{\mathrm{GL}_4(F)}(\tau)$. On the other hand, we can also write π_1 as $I_{\bar{P}_{2,2}}^{\mathrm{GL}_4(F)}(\pi_0)$ with $\pi_0 = \pi_{11} \otimes \pi_{12}$ and π_{1i} be the parabolic induction of $\chi_{2i-1} \otimes \chi_{2i}$.

Theorem C.4.7. *With the same assumptions as in Conjecture C.4.2 and with the notation above, the followings hold.*

1. *If $\bar{P}_{2,2} \subset Q$ and if π_2 is an essentially tempered representation, Conjecture C.4.2 holds.*

2. *If $Q \subsetneq \bar{P}$ and if $\pi' = \pi_{11} \otimes \pi_{12} \otimes \pi_2$ satisfies the condition (40) in [L01], Conjecture C.4.2 holds.*

As in the Ginzburg-Rallis model case, the assumption on Q in Theorem C.4.7(2) can be removed if we can prove the holomorphic continuation of certain generalized Jacquet integrals (i.e. the hypothesis in Section 14.3).